Case Study: Population Modeling

Animation: Population Modeling **To get you going on the Case Study!**

In a previous section we have discussed how to set up scientific models using differential equations. We have also discussed methods of solving the equations either exactly or numerically. Now that we have some experience with exact techniques such as separation of variables and approximate techniques such as Euler's method, we are going to consider a *Case Study in Calculus* (CSC) that will require us to find both exact and numerical solutions.

As we have said before, the purpose of a CSC is to consider a real application of calculus, with real data. In this section, we will study two approaches to modeling populations in order to:

Objective: Predict the size of the US population well into the 21^{st} century.

In a CSC, we adopt the viewpoint of a mathematician who works in a setting where he or she is expected to find answers to real, everyday questions like the one above. In this particular CSC, a possible scenario is that you are a mathematician working for the Federal Census Bureau. Because the nation needs to know population trends so that it can better predict and plan for future impact on resources, your task is to provide population estimates.

An important purpose of this CSC is to give first-hand experience doing mathematics in an experimental setting. As such, you should keep in mind the **Scientific Method** which in our context takes the form:

1. Translate real-world problems into mathematical models.
2. Subject the models to mathematical analysis and prediction.
3. Draw conclusions from the models.
4. Test the conclusions in the laboratory and compare the results with the original real-world data.
5. Revise the model as necessary and repeat the above steps until the model is a reliable predictor of real-world observations.

There are two approaches below, called *Malthus* and *Verhulst*, modeling the U.S. population based on two different assumptions. The experimental and analytical activities are very parallel, carrying out the same set of investigations, but with the two different models. For each model, there will be a Setup, and a Thinking and Exploring step to complete. But there will be only one final written summary that combines and compares the conclusive interpretations for the two approaches.

The Malthus and Verhulst Models

The Malthus model for growth of a population assumes an ideal environment. Resources are unlimited, disease is constrained, and individuals are happy. The population increases at a rate proportional to the number of individuals present.

In reality populations do not exhibit such unrestricted growth since the density of individuals eventually rises until conditions for living are no longer satisfactory and competition with other individuals in the same population or with other organisms reduces fertility and longevity of the population. Although the initial stages of population growth seem to be exponential, the growth curve eventually levels out and approaches a horizontal asymptote that represents the maximum carrying capacity of the environment. Thus, the Verhulst model, that takes into account the effects of a limiting environment, is a more realistic model.

The CSC will study each of these models in turn and compare them for the US census data. We will use only the populations recorded in the census of 1790 and of 1990.

USA Census (in millions)	
Year	Population
1790	3.9
1990	250

The Malthus Model: Exponential Growth

In this model, we assume uninhibited exponential growth of the population. Thus, starting with a population of 3.9 million in 1790, we have the Initial Value Problem

$$\frac{dQ}{dt} = kQ, \qquad Q(0) = 3.9$$

An explicit statement of our goal is as follows:

Objective: Using actual U.S. population census data, find the growth constant k of the U.S. population during the period 1790 to 1990, assuming exponential growth.

A major purpose of the CSC is to learn to think clearly about such applied problems. As an aid, we have structured the steps of the analysis in the form of a report that you will complete. The main sections of the report are as follows (see homework). We will reach our objective in two different ways, which we describe in the Setup.

Setup: This is what is called the *modeling phase* where we set up the differential equation. We have already done that above. We have written the initial-value problem that models the U.S. population beginning in the year 1790, assuming exponential growth. The initial condition reflects the actual U.S. population (in millions) in the year 1790, which for the sake of this investigation we take to be 3.9 million persons.

In the Thinking and Exploring part below, you will be using Euler's method for solving the IVP we just described. You will carry out a trial and error experiment using several values of the growth constant k seeking a value that leads to a projected population in 1990 that is close to 250 million. Once the experimental process is complete, you then will solve the IVP explicitly by the method of separation of variables and compare this value of k with the experimental one.

We have described what kind of mathematical facts you will be looking for, and the methods that we expect you to use. You are not to carry out the investigation as part of your setup since this is part of Thinking and Exploring below.

Thinking and Exploring: First, use Euler's method to find k experimentally so that the population in 1990 is 250 million people. Then use the result to predict the population in 2100. Finally, solve the IVP by the method of separation of variables and find k explicitly.

Applet: Euler Population Predictions **Try it!**

The Verhulst Model: Limited Exponential Growth

Various alternative models of population growth have been suggested. In most of them the growth rate of the population is assumed to be dependent on population size rather than constant. One especially well-known model was proposed by Verhulst in 1838 to describe the growth of human populations. The same model was independently used by Pearl and Reed (1920) to describe the growth of the U.S. population. The Verhulst model assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support. Specifically, it replaces the growth constant k by the expression

$$k\frac{M - Q(t)}{M}$$

Note that when $Q(t)$ is small (relative to M) this expression is close to k, and as $Q(t)$ approaches M it becomes close to 0. This leads to the differential equation

$$\frac{dQ}{dt} = k\frac{M - Q}{M}Q$$

The factor $\frac{M-Q}{M}$ that has a value between 0 and 1 is sometimes called the *unrealized potential for population growth*. When Q is small it has a value close to 1, and the growth of the population is essentially exponential. As Q approaches its asymptotic limiting value, however, the factor $\frac{M-Q}{M}$ is close to zero, and the population grows ever more slowly.

Here is the specific objective we will be pursuing during the Verhulst part of the CSC, followed by the setup, and Thinking and Exploring.

Objective: The U.S. population cannot sustain exponential growth indefinitely. The Malthus model gives unrealistic projections of the population over the next century. We would like to use the Verhulst model instead to make such projections. Some demographers and environmental analysts have proposed $M = 750$ million as the asymptotic limit of the U.S. population. Accepting this value, we would like to find the constant k in the Verhulst equation that models the growth of the population from 1790 to 1990, and then project the model over the next several centuries.

Setup: The differential equation is given above. We also need to assume that $Q(0) = 3.9$ million, and $M = 750$ million, the maximum value of the population $(0 \leq Q(t) \leq M)$. The rest of the setup is the same as the setup for Malthus. That is, under Thinking and Exploring below, you will be using Euler's method to find k experimentally so that the population in 1990 is 250 million, and then you will be solving the differential euqation explicitly (by separation of variables) and comparing the two values of k.

Thinking and Exploring: Repeat the steps that you carried out for Malthus under Thinking and Exploring. That is, first implement Euler's Method for the Verhulst equation, and carry out the actual experimentation of running the program with different values of the constant k. Your objective, as before, is to match the known U.S. population of 250 million in the year 1990. Next, using this value of k, project the U.S. population, using the Verhulst model, in the year 2100, the year 2200, and for a few centuries beyond that. Finally, show how the same problem can be handled algebraically, using the mathematical solution of the IVP (by separation of variables). Project the population in the future year using the algebraic methods. How do they compare with the values obtained experimentally using Euler's method?

Note: In the process of solving the Verhulst differential equation by separation of variables, you will need the decomposition

$$\frac{M}{(M-Q)Q} = \frac{1}{M-Q} + \frac{1}{Q}$$

Verify for yourself that this equation is correct. The right-hand sum is easy to rewrite as the fraction on the left-hand side of the equation. But it is not so obvious that the fraction on the left can be written as the sum on the right if that sum is not known. To decompose the fraction on the left into the sum on the right is the subject of a method called *partial fractions*. It is a general method of integration that is taken up in a second course in calculus. We need only this instance of its use here. In fact, given the decomposition we need not concern ourselves further where it comes from because we can verify independently that the equation is correct. Just show that the right-hand side equals the left.

Combined Comparative Interpretation and Summary

This part of the report will be comparative in nature. Now that you have derived mathematical facts, based on the Malthus and Verhulst models, it is time to interpret and summarize the mathematical results in terms of the original objective to project the U.S. population into the future, based on the two models, and to learn as much as possible from the exercise that might guide public policy in the coming decades.

The written summary is a very important part of the report because mathematicians often have to explain their work to non-mathematicians, and be convincing about the proposed course of action. Pretend that your synopsis is going to appear in the next issue of a magazine such as *Scientific American*. Include enough details so that a reader would learn what the major issues of the report are, and how you went about addressing them. You should take care to write in complete sentences using correct rules of standard English grammar.

You will probably want to comment on how realistic the Malthus and Verhulst models seem to be. Explain why you think the models do, or do not, realistically predict the U.S. population very far into the future. What environmental or social factors might influence the growth of the population, diminishing the usefulness of the Malthus model? What advice do you have for public policy makers? Make the report interesting, compelling. Ask yourself: Would a policy maker be able to understand it, and feel compelled to follow its recommendations?

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**