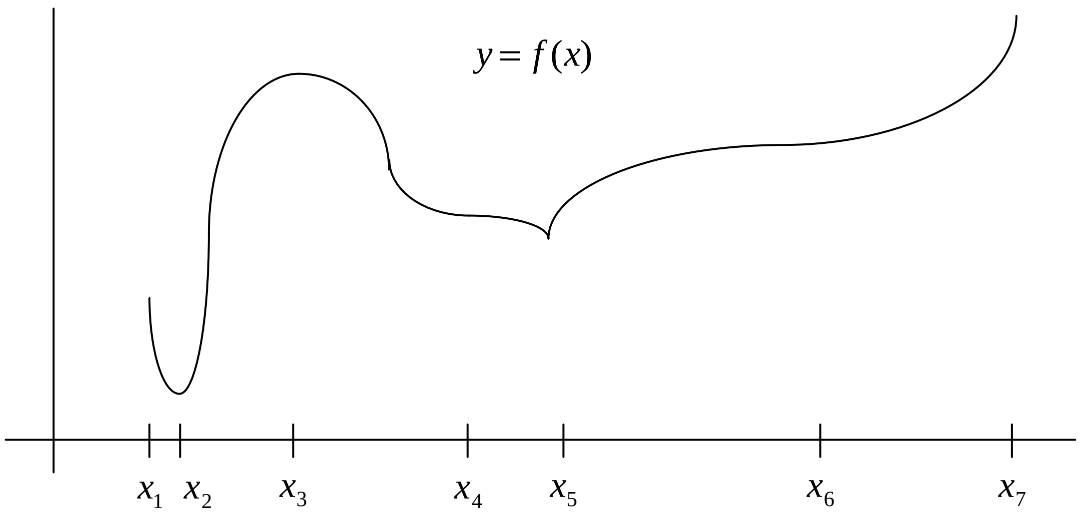
**Issues in Curve Sketching**


One of the most useful applications of the derivative is in curve sketching. Roughly speaking, given a function defined by a formula, we want to produce its sketch. As we shall see, the first and second derivative are excellent tools for this purpose.

## The First Derivative and Extreme Values

Here is the graph of a function:




The function is defined on the interval $[x_1, x_7]$. In what follows, we want to develop the language to describe the high points and the low points of the graph, as well as the general shape. We will start with a discussion of the various kinds of extreme values. Don't be put off by the number of definitions in this section. They give us a vocabulary with which to discuss the concepts, and need to be recorded for easy reference. However, once we get to the examples at the end of the section, you will see that the analyses flow very smoothly.

**Definition 1:** The function $f$ has an absolute maximum value $f(x_0)$ at $x_0$ in its domain if $f(x) \leq f(x_0)$ for all $x$ in the domain. The function $f$ has an absolute minimum value $f(x_0)$ at $x_0$ in its domain if $f(x_0) \leq f(x)$ for all $x$ in the domain.

**Example 1:** In the graph above, the absolute maximum value occurs at $x_7$ and the absolute minimum value at $x_2$. The absolute max and absolute min values are the greatest and least values taken on by the function throughout its domain.

We can see from the graph that the absolute maximum and minimum values do not tell the entire story. There are also local maximum values that correspond to the peaks of the graph, and local minimum values that correspond to the valleys. The concept of *local* is described in terms of neighborhoods.

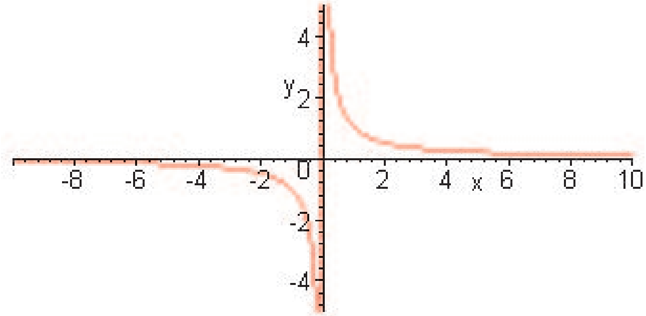
**Definition 2:** A neighborhood of the point $x = x_0$ is an open interval containing $x_0$.

**Definition 3:** The function $f$ has a local maximum value $f(x_0)$ at $x_0$ in its domain if $f(x) \leq f(x_0)$ for all $x$ in the domain of $f$ in some neighborhood of $x_0$. The function $f$ has a local minimum value $f(x_0)$ at $x_0$ in its domain if $f(x_0) \leq f(x)$ for all $x$ in the domain of $f$ in some neighborhood of $x_0$.

**Example 2:** With reference to the graph of the function above, there are local maximum values at $x_1, x_3, x_7$, and local minimum values at $x_2$ and $x_5$. Note that the open interval that defines the local neighborhood need not be wholly contained in the domain of the function; this is true in this example at the enpoints of the domain. In fact, we can define an endpoint in this way.

**Definition 4:** An endpoint of the domain is a point of the domain that does not belong to an open interval contained entirely in the domain.

**Example 3:** The function $f(x) = 1/x$ has no absolute maximum value, and no absolute minimum value. Nor does it have any local maximum or minimum values. The domain has no endpoints.




**Critical Points**

When it exists, the derivative can be used to detect points where local maxima and minima occur.

**Definition 5:** The point $x$ is a critical point if $x$ is in the domain of $f$ and $f'(x) = 0$.

**Definition 6:** The point $x$ is a singular point if $x$ is in the domain of $f$ but $f'(x)$ is not defined.

The following theorem is a consequence of the Mean Value Theorem. Although we will not prove it, the theorem is extremely important because it will lead to a procedure for finding extreme values.

**Theorem 1:** Suppose the function $f$ is defined on an interval $I$ and has a local maximum (or local minimum) value at $x = x_0$ in $I$. Then $x_0$ must be one of the following:

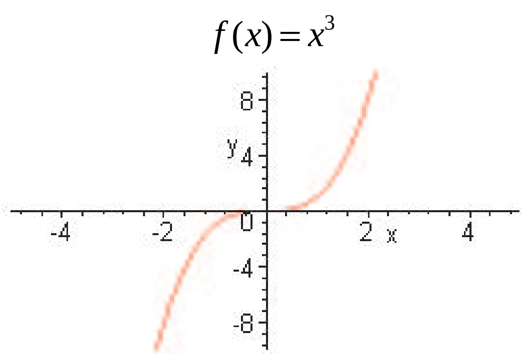
1. $x_0$ is a critical point of $I$ if $f'(x_0)$ exists [i.e, if $f'(x_0)$ exists, then $f'(x_0) = 0$.]; or
2. $x_0$ is a singular point of $f$; or
3. $x_0$ is an endpoint point of $I$.

**Example 4:** Back to the first graph above. The theorem gives a clasification of the points where the local maximum and minimum values occur. Note that $f'(x) = 0$ for $x = x_2, x_3$. There is a singular point at $x_5$ because the derivative does not exist (the limits of the difference quotients from the left and right are not equal, just as with a corner point). And, of course, there are the endpoints.

Our real objective is to start without a graph of a function, and use the first derivative (and the second derivative) to graph it. Thus, we have to be very careful about what our theorems say.

For example, the theorem does not say that if $f'(x) = 0$, then the function has a local maximum or minimum value at $x$. Instead, it tells us that the solutions of $f'(x) = 0$ are candidates only. At $x_6$ in our sketch it looks as though the derivative is zero (horizontal tangent line), but there is neither a local max nor local min there.

**Example 5:** If $f(x) = x^3$, then $f'(x) = 3x^2$. Hence, $f'(0) = 0$. But the function does not have either a local maximum or local minimum value at $x = 0$.



A continuous function on a closed bounded interval will always have an absolute maximum and an absolute minimum value. The proof of this theorem is beyond the scope of our work and lies in the more advanced subject of real analysis. However, it is important, because together with what we already know about local extrema, it gives us a procedure for finding absolute extreme values on a closed bounded interval.

**To find the extreme values (absolute maximum and absolute minimum values) on a closed bounded interval:**

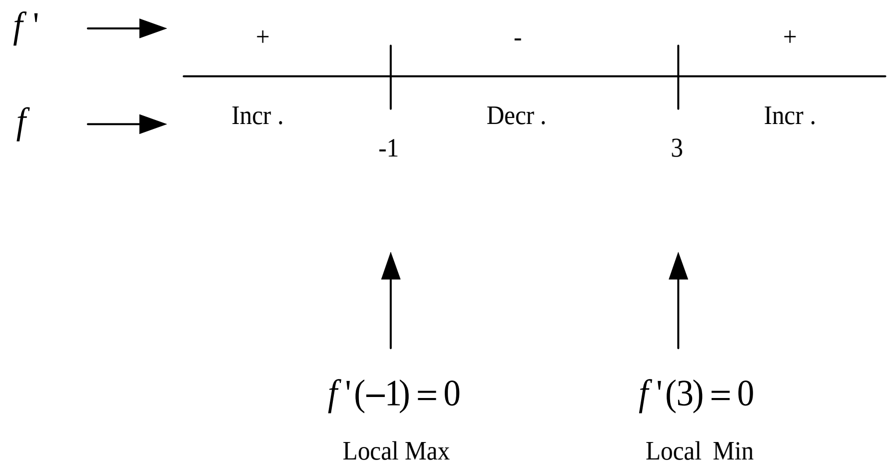
1. Find the critical points (i.e., solve for $x$ in $f'(x) = 0$).

2. Find the singular points (i.e., points $x$ for which $f'(x)$ is not defined).

3. Test the points in 1 and 2, and test the endpoints. The maximum and minimum values will be among them.

**First Derivative Test for Local Max/Min:** In Example 5 we saw that $f'(x_0) = 0$ does not imply that $f$ has a local max or local min value at $x_0$. However, suppose that $x_0$ is an interior point of an interval and $f'(x_0) = 0$. Then if on a neighborhood of $x_0$, we have that $f'$ is positive to the left of $x_0$ and negative to the right, then this means that $f$ is increasing as we approach $x_0$ from the left and decreasing as we continue past $x_0$ to the right. Hence, the function $f$ has a local maximum at the point $x_0$. This is the so-called first-derivative test. (Analogously, if$f'$ is negative to the left and positive to the right of $x_0$, then $f$ has a local minimum at $x_0$.

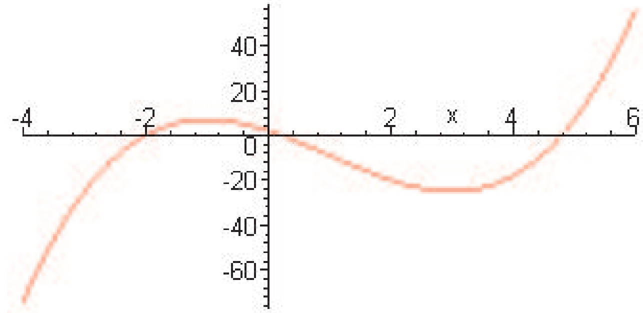
**Example 6:** Find the maximum and minimum (both local and absolute) values of the function $f(x) = x^3 - 3x^2 - 9x + 2$ on the interval $[-2, 2]$. To solve the problem, we note first that there are no singular points because the function is a polynomial. Hence, we will proceed to find the critical points and to determine which ones give local maxima and minima. If we display a sign table for $f'$, we will be aided in using the first derivative test. Here is the derivative: $f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$.



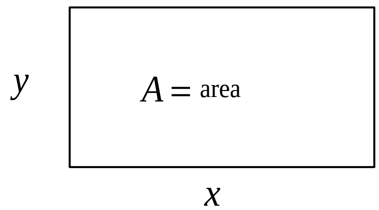

Note that because the function is increasing and then decreasing as it passes through the critical point at $x = -1$, we have a local maximum value there. Analogously, we have a local minimum value at $x = 3$. To find the maximum and minimum values of $f$ on the interval, we make a table of values containing the critical points and the endpoints. Note that there is only one critical point in the interval.

| $x$ | $f(x)$ |
|---|---|
| -2 | 0 |
| -1 | 7 |
| 2 | -20 |

So, the maximum value is 7 and the minimum value is -20 on the interval $[-2, 2]$. Also note that $f(3) = -25$ would be the minimum value on any interval that included 3. [Note: Here is a graph of the function that you can produce on your calculator. We still need some information about the shape of the graph to do as well by hand. We will solve that problem next by using the 2nd derivative.]




**Example 7:** (Optimization) What dimensions maximize the area of a rectangle of fixed perimeter 1 meter? To solve this problem, let's assign the variables as in the following sketch:

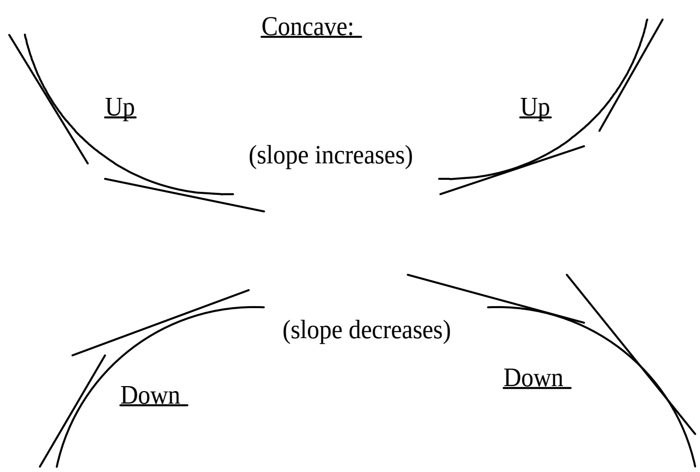



Then $2x + 2y = 1$ implies $y = 1/2 - x$. Thus, $A = xy = x(1/2 - x) = x/2 - x^2$. Now, we calculate the derivative, set it equal to 0, and solve: $A' = 1/2 - 2x$; so, $1/2 - 2x = 0$ gives $x = 1/4$. If we argue that $x$ can be any value in the interval $[0, 1]$, with both endpoints representing degenerate rectangles of no width or no height, and hence of 0 area, then the area is a maximum when $x = 1/4$. That is, when the rectangle is a square, 1/4 meter on a side.

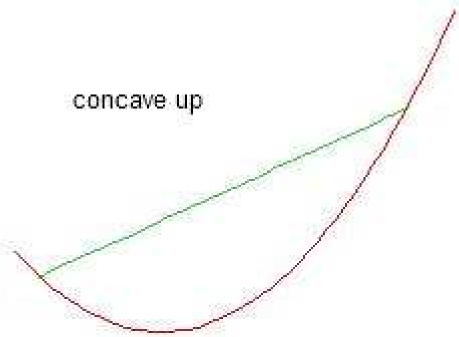
**Applet:** Curve Sketching: Increasing/Decreasing **Try it!**

### The Second Derivative: Concavity and Inflection Points

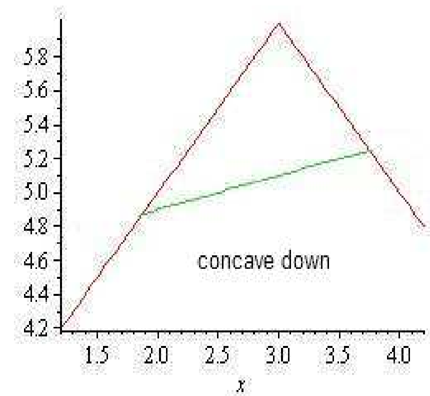
Suppose $y = f(x)$ is a given function. If $f'$ is an increasing function on an open interval, then the slope of the tangent line to the graph of $f$ is increasing as we move from left to right. The graph of $f$ thus bends upward. Similarly, if $f'$ is decreasing, then the graph bends downward.




Observe that in the sketches above, the chords lie above or on the graph of the function when the graph bends upward (and below or on when the graph bends downward). Indeed, we will turn the latter observation into a definition in a more general setting, and we will call the graph *concave up* (or *concave down*).




**Definition 7:** The function $f$ is concave up on an open interval if $f$ is continuous there and all the chords to its graph lie above or on it. Similarly, a continuous function on an open interval is concave down if all chords to its graph lie below or on it.




Note that the function $f(x) = |x|$ is continuous and concave up on its domain. We can conclude this by recalling its V-shaped graph. If however in addition to continuity, we make one mare assumption about $f$, we have a useful criterion to apply.

**1st Derivative Test for Concavity**: Assume that $f'$ exists on an open interval and is increasing, then $f$ is concave up. Similarly, $f$ is concave down on an open interval if $f'$ exists and is decreasing there.

We saw this above when we first introduced the concept of concavity. For functions with derivatives, we will see shortly that the derivative becomes a powerful tool for analyzing the shape of the function's graph. But first let's define a further concept related to concavity. If you try putting together pieces of the graphs from the above sketches, you can see that at a point where $f'(x) = 0$, the concavity can either remain the same as you move from left to right or it can switch from up to down, or from down to up. We give a special name to the latter situation.

**Definition 8:** The continuous function $f$ has an inflection point at the point $x_0$ if the concacity switches at $x_0$ from up to down or down to up.

From the fact that a positive second drivative on an interval implies that the first derivative is increasing, we can use $f''$ to test for concavity as in the next theorem.

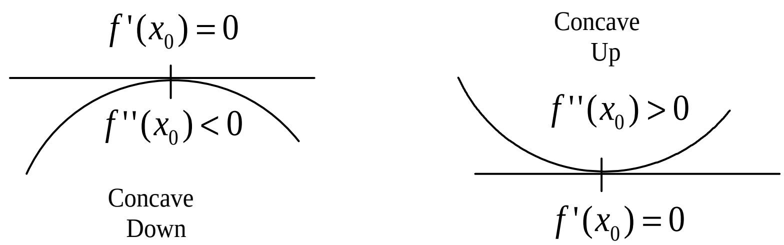
**Theorem 2:** (2nd Derivative Test for Concavity)

1. If $f''(x) > 0$ on an interval, then $f$ is concave up on the interval.
2. If $f''(x) < 0$ on an interval, then $f$ is concave down on the interval.
3. If $f$ has an inflection point at $x_0$ and $f''(x_0)$ exists, then $f''(x_0) = 0$.

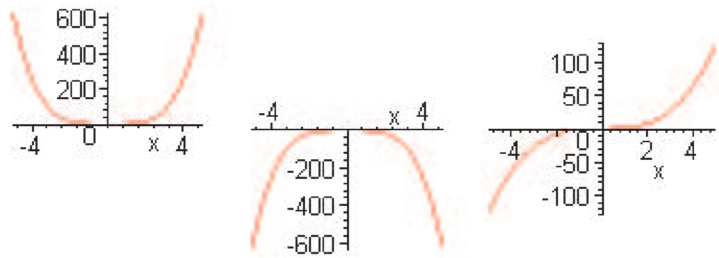
Closely related is the so-called *Second Derivative Test* for local max/min.

**Theorem 3:** (2nd Derivative Test for Local Max/Min.)

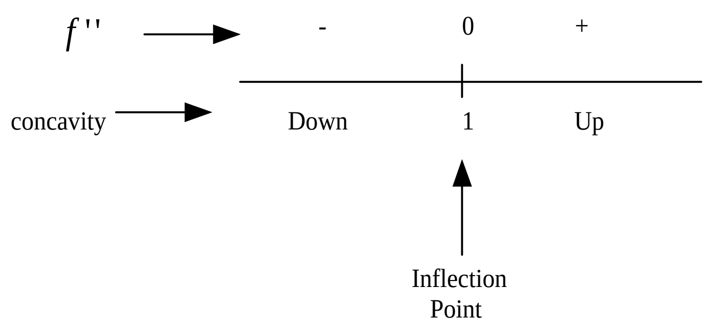
1. If $f'(x_0) = 0$ and $f''(x_0) < 0$ then $f$ has a local maximum at $x = x_0$.
2. If $f'(x_0) = 0$ and $f''(x_0) > 0$ then $f$ has a local minimum at $x = x_0$.
3. If $f'(x_0) = 0$ and $f''(x_0) = 0$ then no conclusion can be drawn. ($f$ may have a local minimum, a local maximum, or an inflection point.)



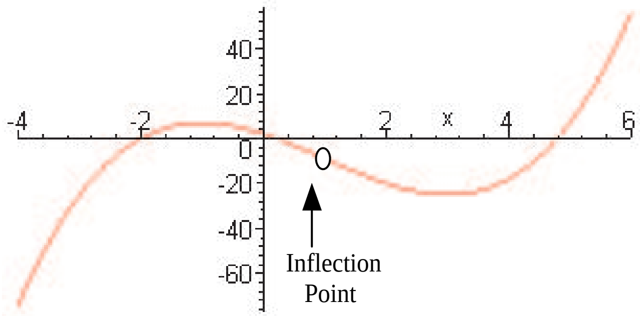

**Example 8:** To illustrate part 3 of the theorem, note that all three of the following functions satisfy the conditions $f'(x_0) = 0$ and $f''(x_0) = 0$ at $x_0 = 0$: $f(x) = x^4$, $f(x) = -x^4$, $f(x) = x^3$. But $x^4$ has a minimum at 0; $-x^4$ has a maximum at 0; and $x^3$ has neither at 0.




**Example 9:** (Example 6 continued) Find the intervals over which the function $f(x) = x^3 - 3x^2 - 9x + 2$ is concave up, and those where it is concave down. Also, find all points of inflection (if any). We need the second derivative: $f'(x) = 3x^2 - 6x - 9$; $f''(x) = 6x - 6 = 6(x - 1)$. Note that $f''(x) = 0$ implies that $x = 1$ is a candidate for an inflection point. We make a sign table for $f''$.




Hence, because the concavity switches at $x = 1$, this is indeed an inflection point. Likewise, the graph is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.



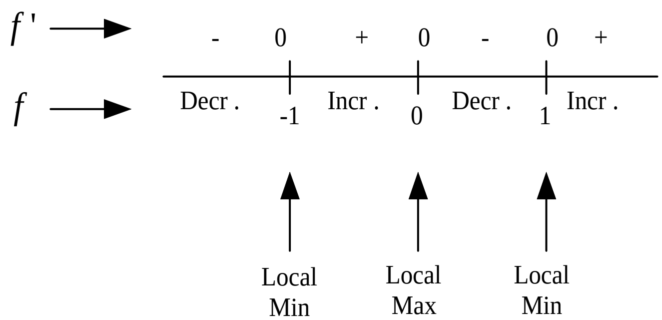

**Applet:** Curve Sketching: Concavity **Try it!**

**Curve Sketching with $y'$ and $y''$: Putting it all together**

We now have enough techniques in hand to sketch the graph of a function using the first and second derivative. This is the goal of the section, and we have finally reached it. Observe in the following examples how straightforwardly the analysis proceeds. But without the foregoing vocabulary and essential ideas, we would not know how to begin. Now we do.

**Example 10:** Sketch a graph of the function $f(x) = x^4 - 2x^2 - 3$ using $f'$ and $f''$. We start by computing the first and second derivatives: $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$; $f''(x) = 12x^2 - 4 = 4(3x^2-1)$.

Increasing/Decreasing: we make a sign table for $f'$.




Concavity: we make a sign table for $f''$.

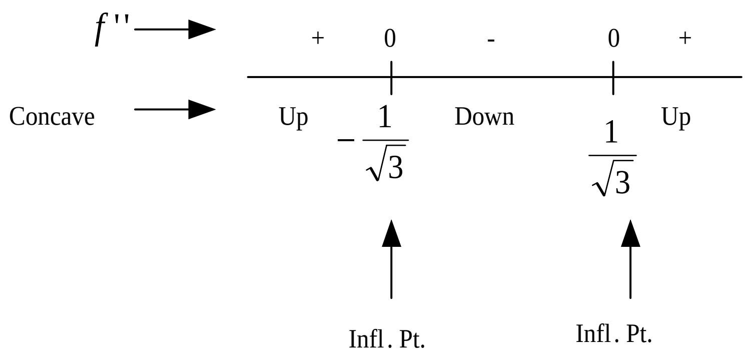
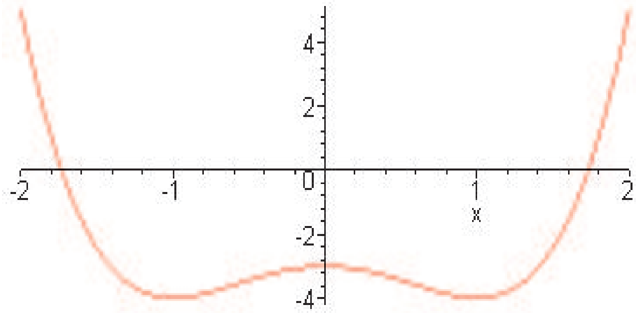



Table of Values:

| $x$ | $f(x)$ |
|---|---|
| -1 | -4 |
| $-1/\sqrt{3}$ | -32/9 |
| 0 | -3 |
| $1/\sqrt{3}$ | -32/9 |
| 1 | -4 |

Sketch: (I.e., assemble the above information into a sketch.)




**Example 11:** Sketch the rational function $f(x) = (x^2 - 1)/(x^2 - 4)$. Find all vertical and horizontal asymptotes.

Zeros: $f(x) = 0 \Rightarrow x^2 - 1 = 0$, or $x = \pm 1$.

Vertical Asymptotes: $f(x) = \frac{(x-1)(x+1)}{(x-2)(x+2)}$ implies vertical asymptotes $x = 2$ and $x = -2$. The limit from the left at $x = 2$ is $-\infty$ and the limit from the right is $\infty$. The function is symmetric about the y-axis; so the limit from the left at -2 equals $\infty$ and the limit from the right at -2 is $-\infty$.

Horizontal Asymptotes: $\lim_{x\to\infty} f(x) = 1$. By symmetry, $\lim_{x\to-\infty} f(x) = 1$. Thus, $y = 1$ is the horizontal asymptote.

Increasing/Decreasing:

$$f'(x) = \frac{(x^2-4)2x - (x^2-1)2x}{(x^2-4)^2} = \frac{2x(-3)}{(x^2-4)^2} = -\frac{6x}{(x^2-4)^2}$$

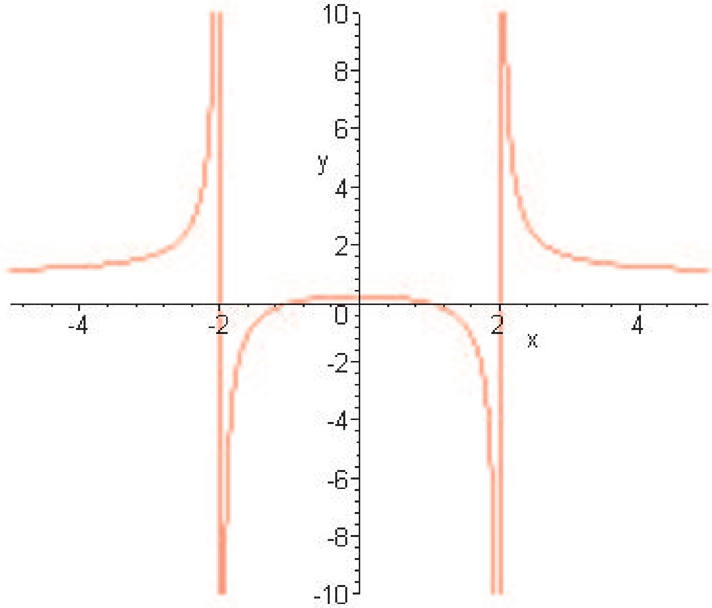
Now, $f'(x) > 0 \Rightarrow -6x > 0$, or $x < 0$. Thus, $f$ is increasing on the interval $(-\infty, 0)$. Similarly, $f$ is decreasing on the interval $(0, \infty)$.

Concavity:

$$\begin{aligned} f''(x) &= \frac{(x^2-4)^2(-6) - (-6x)2(x^2-4)2x}{(x^2-4)^4} \\ &= \frac{(-6)(x^2-4)(x^2-4-4x^2)}{(x^2-4)^4} \\ &= \frac{6(x^2-4)(3x^2+4)}{(x^2-4)^4} \\ &= \frac{6(3x^2+4)}{(x^2-4)^3} \end{aligned}$$

Now, $f''(x) = 0$ has no solutions. Hence, there are no candidates for points of inflection. Note that the numerator of $f''(x)$ is always positive. Thus, the sign of $f''$ comes from the denominator. A check shows that $f''(x) > 0$ on the intervals $(-\infty, -2)$ and $(2, \infty)$ and hence $f$ is concave up there, and $f''(x) < 0$ on the interval $(-2, 1)$ implying that $f$ is concave down there. Moreover, -2 and 2 are not inflection points because they do not belong to the domain of $f$.

Here is a sketch:




**Exercises:** Problems **Check what you have learned!**
**Videos:** Tutorial Solutions **See problems worked out!**