Slope Fields and Euler's Method

In this section we are going to study the geometric information that we get from a differential equation that gives an explicit formula for the derivative. Our intent is to use that information to find a solution of the equation. Consider the differential equation

$$\frac{dy}{dx} = F(x, y); y(x_0) = y_0$$

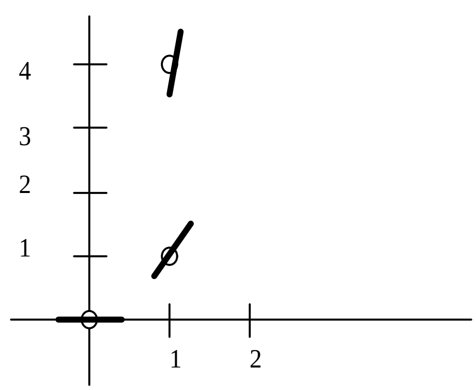
where $F(x, y)$ is a given function of x and y. (For example, $F(x, y)$ might be $8\sqrt{y}$, or it might be $x - y$.) The problem is actually stated in the form of an Initial Value Problem (IVP). We are looking for the particular solution of the equation that passes through the point (x_0, y_0). Assume now that we do not know the solution $y(x)$ and let us interpret what the equation tells us about tangent lines.

The equation says that at any point (x, y) in the plane we can compute the slope $\frac{dy}{dx}$ of the tangent line through that point. That is, at each point (x, y) in the plane, we can draw a short straight line whose slope is $F(x, y)$ from the differential equation. The resulting two-dimensional plot of tangent lines is called the *slope field* or *direction field* of the differential equation.

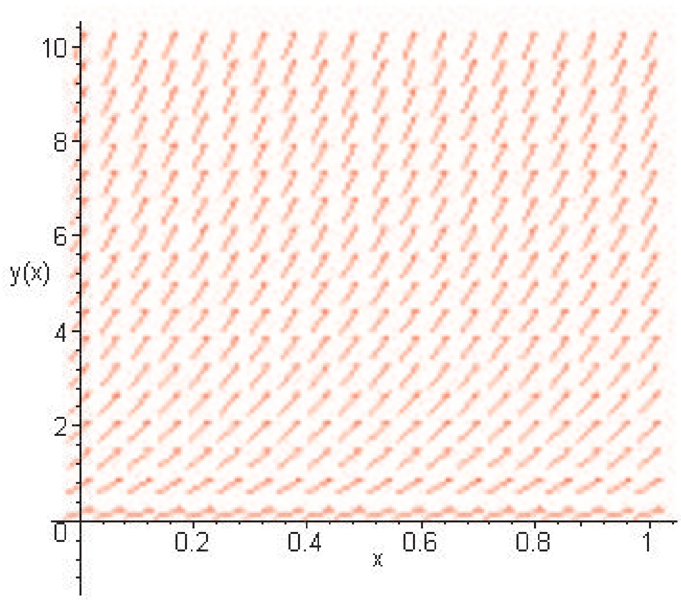
Slope fields are important because sometimes we can guess the shape of the solution curve by sketching a curve that satisfies the given initial condition and follows the slopes of the slope field.

Example 1: Let $F(x, y) = 8\sqrt{y}$. Then we can make a table of points (x, y) and corresponding slopes given by the differential equation $\frac{dy}{dx} = F(x, y)$.

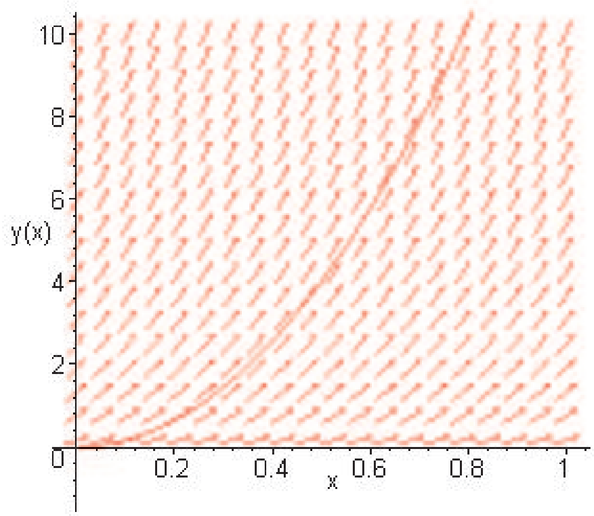
Point (x, y)	Slope $F(x, y)$
$(0, 0)$	0
$(1, 1)$	8
$(1, 4)$	16



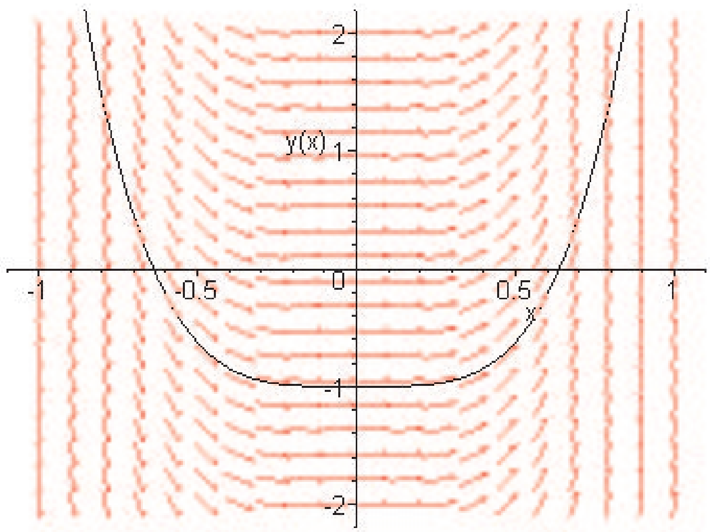
We see from the above example that this can be an exhaustive procedure to carry out by hand. However, it is an excellent task for a computer. Here is a computer display of a slope field for the same equation $\frac{dy}{dx} = 8\sqrt{y}$.



Can you guess the shape of the solution curve that passes through $(0, 0)$? Put your pencil at $(0, 0)$ and see if you can sketch in a curve that *follows the slopes*. That is, when you are done, each line that touches your curve should look like the tangent line at that point.



Example 2: Here is another example of using slope fields to visualize solutions of differential equations. We will consider the differential equation $\frac{dy}{dx} = 24x^3$. The slope field shows a plot of slopes for this equation. The particular solution plotted here is seen to be the curve that passes through the initial point $(0, -1)$ and follows the slope field. (By the way, you can solve the equation analytically. What is the general solution?)



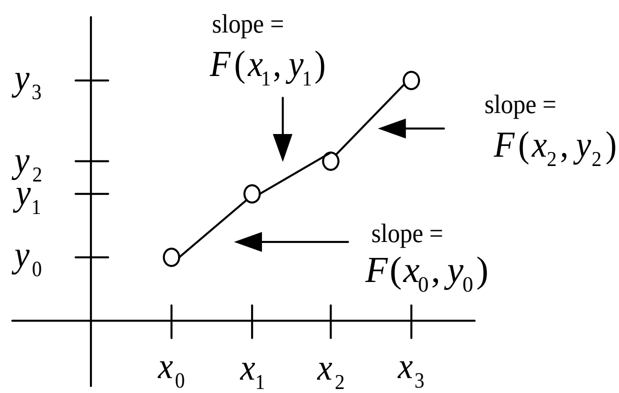
The Next Step: Euler's Method

The above examples suggest a simple way to approximate the desired particular solution numerically. Since the differential equation determines the slope at each point (x, y) of a particular curve, we can approximate a nearby point (x_1, y_1) on the curve by following the tangent line. The resulting procedure is called *Euler's Method*.

Assume that the following IVP is given:

$$\frac{dy}{dx} = F(x, y); P_0 = (x_0, y_0)$$

The method consists of starting at the initial point $P_0 = (x_0, y_0)$, specifying an increment Δx, and plotting a sequence of line segments joined end to end. The slope of each segment is the value of the derivative at the initial point of the segment. We then use the polygonal path as an approximation to the graph of the solution curve through P_0.



Note that the x-coordinates of the points are equally spaced Δx units apart. So, for each n, $x_{n+1} = x_n + \Delta x$. It turns out that we can write a formula for y-coordinates as well.

Theorem 1: Given the Initial Value Problem $\frac{dy}{dx} = F(x, y); P_0 = (x_0, y_0)$, and Δx specified, then the endpoints of the line segments that make up the polygonal path in Euler's Method are

$$\begin{aligned} x_{n+1} &= x_n + \Delta x \\ y_{n+1} &= y_n + \Delta x F(x_n, y_n) \end{aligned}$$

where $n = 0, 1, 2, 3, \ldots$.

The proof is not too difficult. For the nth line segment has equation $y_{n+1} - y_n = F(x_n, y_n)(x_{n+1} - x_n)$. Thus, $y_{n+1} = y_n + F(x_n, y_n)(x_{n+1} - x_n) = y_n + F(x_n, y_n)\Delta x$.

Example 3: Let $\frac{dy}{dx} = x - y; y(0) = 1$. On the interval $[0, 1]$ approximate $y(1)$ with two steps of size $1/2$. Here $F(x, y) = x - y$ and $\Delta x = 1/2$. Thus, $y_1 = y_0 + \Delta x F(x_0, y_0) = 1 + (1/2)(-1) = 1/2$; and $y_2 = y_1 + \Delta x F(x_1, y_1) = 1/2 + (1/2)(0) = 1/2$. Therefore $y(1) \approx 1/2$.

The simplicity of the Euler's Method idea is deceiving. The theorem tells us to start at the initial point and step along successively computing the endpoints of the line segments of the polygonal path. The method and its numerical cousins turns out to be one of the most useful and powerful techniques for exploring solutions of differential equations when exact solutions are too difficult or impossible to obtain. The fact that it may be tedious to generate the points by hand is irrelevant. All we need to do is to call in a Computer Algebra System (e.g., Maple, Mathematica, Derive) or an applet for reinforcements. Then we can decrease the size of Δx and get a better approximation of a desired value of y with very little trouble.

Error estimate: If in general, we use Euler's method to approximate the solution curve on the interval $[x_0, b]$, then the approximation of the value of the solution at b gets better as $\Delta x \to 0$. The error, in fact, at b decreases proportionally to $|b - x_0|\ \Delta x$. Thus, the approximating values approach the solution value as Δx goes to 0.

Applet: Euler's Method **Try it!**

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**