**Separable Differential Equations**
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We have already seen that the differential equation $\frac{dy}{dx} = ky$, where $k$ is a constant, has solution $y = y_0 e^{kx}$. We have solved this equation in three ways: by guess-and-check in Section 3.1, and by algebraic manipulation and integration in Section 3.2. The differential equation, representing exponential growth or decay, is also an example of a *separable* differential equation, which we solved as such in Section 3.1.

As introduced in Section 3.1, a first-order differential equation in $x$ and $y$ is called *separable* if it is of the form

$$\frac{dy}{dx} = g(x)h(y)$$

where $y = f(x)$. That is, when the equation is written in terms of differentials, the $x$'s and $dx$'s can be put on one side of the equation and the $y$'s and $dy$'s on the other in such a way that we can solve the equation by integrating both sides:

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

This procedure to solve the differential equation is called the *method of separation of variables.*

**Example 1:** As a review, let's again solve the equation $\frac{dy}{dx} = ky$ by the method of separation of variables. The method begins by rewriting the equation using differentials. First, we separate the $y$'s and $dy$'s from the $x$'s and $dx$'s, and then we integrate both sides of the rewritten equation, and solve for $y$:

$$\frac{1}{y}\,dy = k\,dx$$

$$\int \frac{1}{y}\,dy = \int k\,dx$$

$$\ln|y| = kx + C$$

From this point on, we do exactly what we did before: we solve for $y$ by exponentiating both sides:

$$|y| = e^{kx+C}$$

$$y = \pm e^C e^{kx} = y_0 e^{kx}$$

**Justification for the Method of Separation of Variables:** But why is the method of separation of variables valid? After all, on the left side of the separated equation we are integrating with respect to $y$, and on the right side with respect to $x$. Using differentials facilitates the method and is a reflection of the genius of Leibniz who believed that the notation should be chosen to motivate the correct answer. However, we have just described a subtlety that we don't want to slide over. The method does indeed give the correct answer, but we must prove it. *Proof by notation* will not siffice.

In fact, we need to show that given the equation

$$\frac{dy}{dx} = g(x)h(y)$$

the antiderivative of $\frac{1}{h(y)}$ as a function of $y$ equals the antiderivative of $g(x)$ as a function of $x$.

The function $y = f(x)$ is a solution of the above equation implies that

$$\begin{aligned} f'(x) &= g(x)h(f(x)) \\ \frac{f'(x)}{h(f(x))} &= g(x) \end{aligned}$$

Let $H(y)$ be any antiderivative of $1/h(y)$; so $H'(y) = 1/h(y)$. Then applying the chain rule yields

$$\begin{aligned} \frac{d}{dx}H(f(x)) &= H'(f(x))f'(x) \\ &= f'(x)\frac{1}{h(f(x))} \\ &= g(x) \end{aligned}$$

Thus, the solution $y = f(x)$ satisfies the equation

$$H(f(x)) = \int g(x)\, dx$$

However, this is just the result of the method of separation of variables, which is to rewrite the differential equation as

$$\frac{1}{h(y)}\, dy = g(x)\, dx$$

and to integrate both sides (the left side with respect to $y$ and the right with respect to $x$) obtaining an equation of the form

$$H(y) = \int g(x)\, dx$$

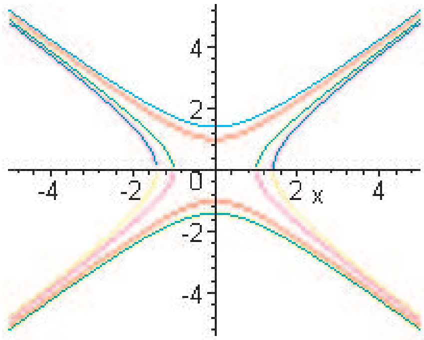
Then this equation implicitly defines the solution $y = f(x)$, as desired.

**More Examples of the Method of Separation of Variables:** In the rest of the section, we will consider additional examples of solving separable differential equations.

**Example 2:** We can use the method of separation of variables to solve the differential equation $\frac{dy}{dx} = \frac{x}{y}$.

$$\begin{aligned} y\, dy &= x\, dx \\ \int y\, dy &= \int x\, dx \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ y^2 - x^2 &= C_1 \end{aligned}$$

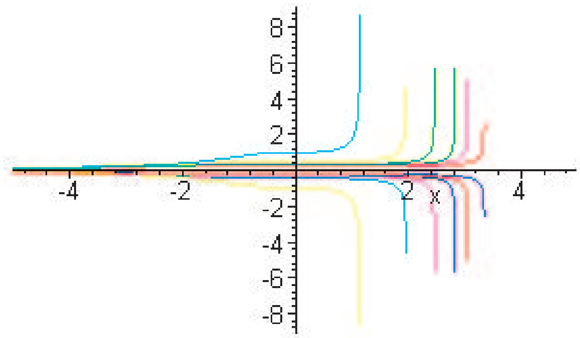
The solution curves are hyperbolas. We can't really go any further unless we knew, say, a point that the solution curve passed through.




**Example 3:** Solve the IVP $\frac{dy}{dx} = x^2y^3; y(3) = 1$. Separating the variables and integrating, we get:

$$\begin{aligned} \frac{1}{y^3}\,dy &= x^2\,dx \\ \int \frac{1}{y^3}\,dy &= \int x^2\,dx \\ -\frac{1}{2y^2} &= \frac{x^3}{3} + C \end{aligned}$$

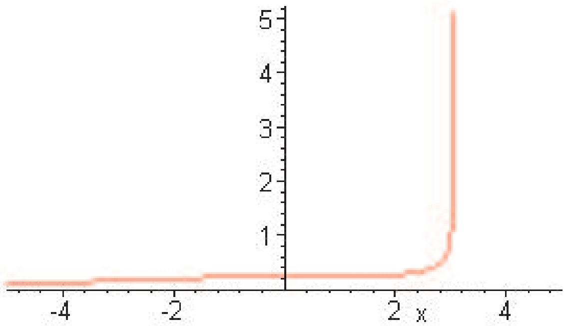
Here are some solution curves:




From $y(3) = 1$, we find the particular solution:

$$\begin{aligned} -\frac{1}{2} &= \frac{27}{3} + C \\ C &= -\frac{19}{2} \\ -\frac{1}{2y^2} &= \frac{x^3}{3} - \frac{19}{2} \\ y^2 &= \frac{1}{19 - \frac{2x^3}{3}} \\ y &= \frac{1}{\sqrt{19 - \frac{2x^3}{3}}} \end{aligned}$$

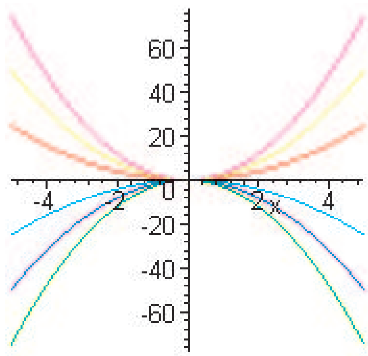
Note that we know that $y$ is the positive square root because we have the initial condition $y(3) = 1$. Here is the particular solution:




**Example 4:** Solve $\frac{dy}{dx} = \frac{2y}{x}$.

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx \\
\ln|y| &= 2\ln|x| + C \\
\ln|y| &= \ln|x^2| + C \\
|y| &= x^2 e^C \\
y &= C_1 x^2
\end{aligned}$$

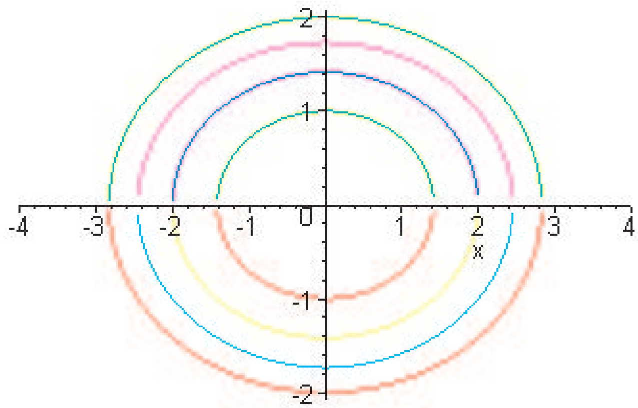
The solution curves are a family of parabolas.




**Example 5:** Solve $\frac{dy}{dx} = -\frac{x}{2y}$.

$$\begin{aligned}
2y\,dy &= -x\,dx \\
\int 2y\,dy &= -\int x\,dx \\
y^2 &= -\frac{x^2}{2} + C \\
2y^2 + x^2 &= C_1
\end{aligned}$$

The solutions are a family of ellipses:




**Example 6:** We can also solve Torricelli's equation by the method of separation of variables. We found in Section 2.18 that the equation is of the form $y' = k\sqrt{y}$, where $k$ is a constant. Then we have:

$$\begin{aligned} y^{-\frac{1}{2}}\,dy &= k\,dx \\ \int y^{-\frac{1}{2}}\,dy &= \int k\,dx \\ 2y^{1/2} &= kx + C \\ y^{1/2} &= \frac{1}{2}kx + C_1 \\ y &= \left(\frac{1}{2}kx + C_1\right)^2 \end{aligned}$$

This is the form of the general solution that we explored in the case study of the previous section.

**Exercises:** Problems **Check what you have learned!**

**Videos:** Tutorial Solutions **See problems worked out!**