

Exponential Growth and Decay

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Derivatives are functions that measure rates of change. A rate of change can be a powerful tool for expressing quantitatively a qualitative description. In the process, we build what we call a model. For example, we know that a nation's population grows or declines depending on the birth and death rates. Can we say something more basic about growing populations? For example, does it make sense to say that at any time t , the rate of change of the size of a growing population is proportional to its size? Probably. In fact, this is most certainly true in many many cases. And look how easy it is to turn this qualitative statement about population growth into an equation.

Just let $y(t)$ be the size of the population at time t . Then the statement becomes:

$$\frac{dy}{dt} = ky; y(0) = y_0$$

where k is a constant of proportionality and $y(0)$ is the size of the population at $t = 0$.

This is another example of a differential equation, that is, an equation that includes derivatives. We have turned it into an *Initial Value Problem* by specifying an initial condition. That is, a particular point $(0, y(0))$ that the solution passes through (in this case, at time $t = 0$). The resulting differential equation becomes a model of the population. Of course, we then have to test the predictions of this model against actual data for this population that we can get from, say, census tables. If the model is a good one, then we can use it to predict the size of the population in the future.

Note that if $k > 0$, then the population is growing, and if $k < 0$, then the population is decreasing. In the next theorem, we solve the differential equation.

Theorem 1: The IVP $\frac{dy}{dt} = ky, y(0) = y_0, k$ constant, has unique solution $y = y_0 e^{kt}$.

The proof is a nice mixture of calculus and algebra. For, if $y \neq 0$, we can rewrite the equation, integrate both sides, and solve for y :

$$\begin{aligned}\frac{1}{y} \frac{dy}{dt} &= k \\ \int \frac{1}{y} \frac{dy}{dt} dt &= \int k dt\end{aligned}$$

The integral on the left should be read as a reversal of the chain rule because $\frac{d}{dt} \ln |y| = \frac{1}{y} \frac{dy}{dt}$. Thus, we can integrate both sides of the integral equation, then exponentiate both sides, and solve for y :

$$\begin{aligned}\ln |y| &= kt + C \\ |y| &= e^{kt+C} \\ &= e^{kt} e^C \\ y &= \pm e^C e^{kt} \\ &= B e^{kt}\end{aligned}$$

where $B = \pm e^C$ is just a constant. Next, we use the initial condition $y(0) = y_0$ to find B : $y_0 = y(0) = B e^0 = B \cdot 1 = B$. Thus, the solution is $y = y_0 e^{kt}$, and the proof is complete.

In applied problems, we go straight to the solution and do not repeat its derivation. Hence, once we have described the model and written down the solution to the IVP from the theorem, the problems involve only algebra.

Example 1: Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 700 cells initially and 900 after 12 hours, how many will be present after 24 hours? To solve this problem, we note first that because the growth is proportional to the number of cells present, then

if we let $y(t)$ be the number of cells present at time t , we know from the theorem that $y(t) = y_0 e^{kt}$ where $y_0 = 700$. So, the problem at hand is to find k from the given information:

$$\begin{aligned} y(12) &= 700e^{12k} \\ 900 &= 700e^{12k} \\ \frac{900}{700} &= e^{12k} \\ \ln\left(\frac{900}{700}\right) &= 12k \\ k &= \frac{\ln 900 - \ln 700}{12} \end{aligned}$$

We can approximate this number using a calculator: $k \approx .0209$. Then we can answer the question: After 24 hours there will be approximately $y(24) = 700e^{.0209 \cdot 24} \approx 1156$ cells.

Doubling Time and Half-Life: In an exponential growth model, the *doubling time* is the length of time required for the population to double. In a decay model, the *half-life* is the length of time required for the population to be reduced to half its size. A characteristic of exponential models is that these numbers are independent of the point in time from which the measurement begins.

Example 2: A radioactive substance that decays according to an exponential model has a half-life of 600 years. What percentage of an original sample is left after 10 years? Once again, our assumption implies that the amount present at time t is given by $y(t) = y_0 e^{kt}$. We use the information about the half-life to find k :

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{600k} \\ \frac{1}{2} &= e^{600k} \\ \ln\left(\frac{1}{2}\right) &= 600k \\ k &= \frac{\ln 1 - \ln 2}{600} \\ k &= \frac{-\ln 2}{600} \\ k &\approx -0.001155 \end{aligned}$$

Now, we can answer the question: $\frac{y(10)}{y(0)} = e^{10k} = e^{10 \cdot (-.001155)} \approx 0.9885$, or 98.85 percent.

Newton's Law of Cooling: This law states that a hot object introduced into an environment maintained at a fixed cooler temperature will cool at a rate proportional to the difference between its own temperature and that of the surrounding environment. That is, if $y(t)$ is the temperature of the object t units of time after it is introduced into a medium at fixed temperature T_m , we have

$$\frac{dy}{dt} = k(y - T_m); y(0) = y_0$$

where k is a constant.

Example 3: Suppose a metal object at 112 degrees Fahrenheit is removed from boiling water and placed on a plate in a room maintained at 68 degrees F. Suppose the object cools to 90 degrees in 5 minutes. How long will it take to cool to 80 degrees? Note that in this problem, $T_m = 68$, and $y_0 = 112$. We will return to the problem after we solve the differential equation.

We need to solve the above differential equation to find $y(t)$. Proceeding as before, we have

$$\begin{aligned}\frac{1}{y - T_m} \frac{dy}{dt} &= k \\ \int \frac{1}{y - T_m} \frac{dy}{dt} dt &= \int k dt\end{aligned}$$

This looks familiar. In fact, because the integrand on the left is the derivative of $\ln|y - T_m|$, we know from following our previous steps that the solution is $y - T_m = Be^{kt}$. Thus, substituting $y(0) = y_0$ yields $B = y_0 - T_m$ and we get the solution

$$y - T_m = (y_0 - T_m) e^{kt}$$

Example 3 (continued): We have that $y - 68 = (112 - 68)e^{kt}$. Thus, from the given information, we find k : $90 - 68 = 44e^{5k}$, so $22/44 = e^{5k}$, and $k = (\ln 22 - \ln 44)/5$, whence $k \approx -.1386294362$. Now, we can answer the question: We want to know at what time t the temperature of the metal object is 80 degrees. That is, we solve for t in the equation $80 - 68 = 44e^{-.1386294362t}$: $12/44 = e^{-.1386294362t}$, or $t = (\ln 44 - \ln 12)/.1386294362$. So, the object will reach 80 degrees approximately 9.37 minutes after it is removed from the water.

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