Modeling with Differential Equations: Introduction to the Issues

A differential equation is an equation involving derivatives and functions. In the last section, we began with a table of values for the rate of change of a function, and we wanted to know how to use it to get information about the function. Now, we are going to assume that we have a complete description of the derivative of a function in the form of an equation that it satisfies, and we are going to address the question: What is the function? That is, how can we obtain f from f' ?

This question is not an issue of idle speculation. Many physical and biological systems can be modeled with differential equations. The main reason is because often it is relatively easy to measure the amount of something that is present at a given time, and then how the amount changes as the system goes from one state to another. For example, we have already discussed the empirical observation that, at any time, the rate of increase of a large rabbit population is proportional to the number of rabbits at that time. If we let $y(t)$ be the number of rabbits at time t, then this observation can be rewritten as the differential equation $\frac{dy}{dt} = ky$, where k is a constant of proportionality. If we let $y(0)$ be the number of rabbits at the beginning of the observation period, then in mathematical terms we say that we have a differential equation and an accompanying initial condition.

In this section we will take up the solution of the general form of this problem. That is, given a differential equation and an initial condition, we want to find the function that satisfies it. To say that f satisfies a differential equation means probably exactly what you think it does; namely, that when f is substituted into the differential equation, the left-hand-side equals the right-hand-side. But this begs the question, how do we find such an f? Many books are devoted to answering this question. A typical one might have a title such as *1001 Methods for Solving Differential Equations*. Our object here is not to become experts in finding solutions to 1001 different kinds of differential equations, but to explore some of the general approaches that always merit consideration.

0.1 Solution by Inspection

It may seem a bit silly, but the first thing to do when examining a differential equation is to try to guess a solution. Once we have a candidate, we then have to verify that our hunch is correct by showing that indeed the function does satisfy the equation. Thus, this method is referred to as *guess-and-check*.

For example, the differential equation

$$\frac{dy}{dx} = ky$$

says that the derivative of the function is a constant times the (same) function. So, we ask ourselves: Is there an elementary function whose derivative is a constant times itself? You might ask, why are we restricting to the elementary functions? This is a good question, and its answer gives one of the reasons that these functions are so important. It turns out that, as we found in the first section, the elementary functions have shown themselves to be extremely valuable in modeling. They come up over and over again. Thus, they are always a good place to start, especially if we are guessing.

But back to answering the question. An exponential function looks like a good candidate. In fact, the derivative of e^{kt} equals ke^{kt}. So, indeed, the derivative of this function is k times the function, and we have solved the equation. See how powerful the guess-and-check method is?

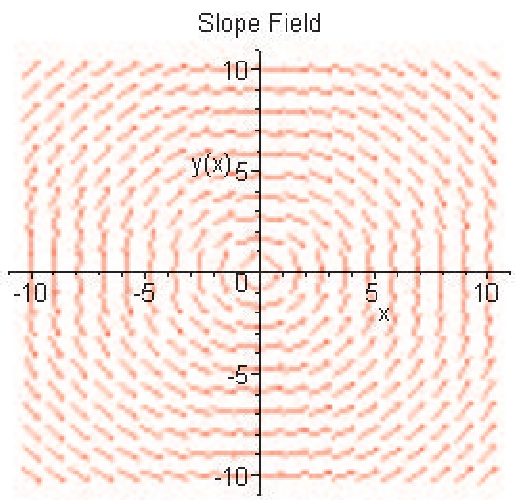
0.2 Slope Fields

In the guess-and-check method, we are considering the equation from a formulaic point of view. That is, we ask ourselves if we can think of a function whose derivative-formula has the desired relationship to the function in the equation. We could also think about the equation from the perspective of slopes. For example, the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

tells us that at every point (x, y) of the plane, the graph of the solution function has slope $-x/y$. For example, at the point (1, 1), the slope of the tangent line to the solution curve passing through that point is -1/1 = -1. Or at (2, -1), the slope of the solution curve passing through it is -2/-1 = 2. This suggests plotting short tangent lines at points of the plane that are sufficiently close together. Then, starting at a given point, draw in the curve that follows the tangent lines as one moves away from that point. We then can try to recognize, if possible, this solution curve as the graph of a function we know. In the worse case, we have approximate values for the solution function at specific values of x.

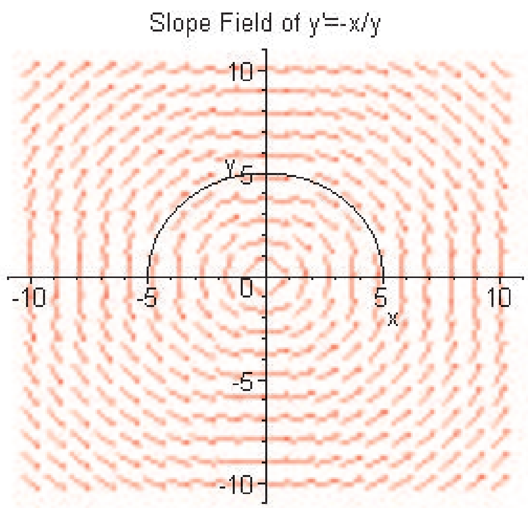
A plot of the tangent lines is called the *slope field* of the differential equation. Here is a slope field for the equation we are considering.



The slope field gives a family of particular solutions. From a starting point, say (-5, 0), it looks like the curve that follows the slopes (or tangent lines) is a semicircle centered at the origin. Thus, although not at first apparent as a guess, the slope field suggests that a solution function might be $y(x) = \sqrt{a^2 - x^2}$, where the curve passes through the point $(a, 0)$. Hence, we substitute this function into the differential equation and check it:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}(-2x) = -\frac{x}{y}$$

Eureka! It checks. Thus, the slope field has given us a candidate to apply our guess-and-check method to. Here is a plot on the slope field of this solution curve in the particular case that it goes through the point (-5, 0).



The slope field raises the issue of how close together we should plot the points. We will return to slope fields later when we will consider what to do if we are unable to guess a solution function whose graph follows the slopes. The technique is called Euler's Method; it is a numerical technique that results almost immediately from what we have just done. The distance between points will be a crucial issue in assuring ourselves of a good approximation to the particular solution we seek. But we will postpone that discussion for now, and continue to concentrate on outlining general approaches to solving a differential equation. In the rest of the Chapter, we will pursue the details and ramifications that we identify as we go along.

Applet: Slope Field **Try it!**

0.3 An Analytical Tool: Separation of Variables

In the last section, we considered Initial Value Problems (IVPs). They involved differential equations that can be put in the form

$$\frac{dy}{dx} = g(x); y(a) = y_a$$

That is, we can solve explicitly for the derivative as a function of x. Hence the general solution is found by integrating both sides of the equation and using the initial condition to determine the particular solution. The IVP

$$\frac{dy}{dx} = x^2 + x + 1; y(0) = 2$$

is such an example. The particular solution is

$$y = \frac{x^3}{3} + \frac{x^2}{2} + x + 2$$

Thus, there is no need to guess; we have a systematic procedure for solving these IVPs.

Differential equations that are just one step removed from these are those that we call *separable*. They are of the form

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

To understand why these equations are called *separable*, we reinterpret them using differentials. Given a function $y = f(x)$, we have defined the differential dy of y according to the formula $dy = f'(x)dx$. Thus, the ratio of dy over dx equals $f'(x)$.

Now, viewing the equation in terms of differentials, a separable equation is one in which we can put all of the y's and dy's (as products) on one side of the equation and all of the x's and dx's (as products) on the other. That is, the variables can be separated to obtain the equation

$$\frac{dy}{h(y)} = g(x) \cdot dx$$

Then, we can integrate both sides of the separated equation, and solve to find a general solution. We will show later in the chapter why the method works, but for now we will confine ourselves to showing how it works. Let's begin by illustrating with an example we have already considered. Let

$$\frac{dy}{dx} = -\frac{x}{y}$$

Separating the variables, integrating, and solving, we get:

$$\begin{array}{lr} y \cdot dy = -x \cdot dx & \text{(separate the variables)} \\ \int y\, dy = \int -x\, dx & \text{(integrate both sides)} \\ \frac{y^2}{2} = -\frac{x^2}{2} + C & \text{(find indefinite integrals)} \\ y^2 = -x^2 + C_1 & \text{(rename constant; } C_1 = 2C) \\ y = \sqrt{C_1 - x^2} \text{ or } y = -\sqrt{C_1 - x^2} & \text{(solve for } y) \end{array}$$

Thus, we see that the solution y to the differential equation satisfies the relationship $x^2 + y^2 = C_1$, where C_1 is a constant; in other words, the points lie on a circle centered at the origin. As soon as we know a point through which the circle passes, we can give a particular solution. When we exhibited above the slope field of this equation, we wanted the solution that passed through (-5, 0). Note that as we could have observed from the slope field, there are two solutions: $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$, the top and bottom halves of the circle.

We can also use the method of Separation of Variables to solve the differential equation $dy/dx = ky$ that we solved above by guess-and-check. Separating the variables, integrating, and solving, we get

$$\frac{dy}{y} = kdx$$
$$\int \frac{1}{y}\, dy = \int k\, dx$$
$$\ln|y| = kx + C$$

Thus, $y = e^{kx+C}$ or $y = -e^{kx+C}$. Hence, $y = e^C e^{kx}$ or $y = -e^C e^{kx}$. Thus, $y = y_0 e^{kx}$, where y_0 is the initial value of the function y.

The method of Separation of Variables is an important analytical tool. Given that the differential equation is in a separable form, the method allows us to approach its solution in a systematic way. It also allows us to put the equation in a form where we can turn to tables or numerical methods to evaluate the integrals if we do not recognize any antiderivatives. But not even every simple-looking equation is separable. For example, consider the equation

$$\frac{dy}{dx} = x - y$$

This equation is not separable. It also is not so obvious what the solution is. Therefore, short of using our generic book *1001 Methods for Solving Differential Equations*, or a computer algebra system to solve the equation, our approach would be to generate its slope field and/or use Euler's method to approximate the solution curve corresponding to a given initial condition.

Applet: Slope Field **Try it!**

0.4 Existence and Uniqueness of Solutions of Initial Value Problems

While we do not want to say much about this subject, we do want to say enough to give assurances that, in most cases we will meet, a solution to an IVP *will exist* and *will be unique*. We will divide the subsection into two parts, first stating the main results, and then outlining additional details for those who would like to see where to look in the advanced literature for even more information. In either case, the material is interesting but comes close to going beyond the scope of this text.

Consider the IVP $y' = F(x, y)$, $y(a) = b$, where $F(x, y)$ is continuous in a domain D that is an open region of the xy-plane and (a, b) is a point in D.

As we have said before, a solution of the IVP is a function $f(x)$ that satisfies the equations; i.e. $f'(x) = F(x, f(x))$ and $f(a) = b$.

In the cases we will meet, the existence of a solution will not be an issue because our techniques will allow us to find one. However, an important result in the theory of differential equations is Peano's Existence Theorem, which states that under the conditions above, there is always at least one solution of the IVP, and any such solution is differentiable. (The theorem actually says more, namely, that any such solution can be extended in both directions to the limits of the region D. In fact, there are maximal and minimal solutions $fMax(x)$ and $fMin(x)$ such that all other solutions lie between them, and the "bundle" of such solutions completely covers the part of the region lying between their graphs. But we will not attempt to explain here exactly what any of this parenthetical comment means.)

Now we come to the main issue about which we need assurances: uniqueness. Even though our techniques will produce a solution, how do we know that it is the only one? In particular, how do we know that there is not another solution different from the one that we get from applying the method of separation of variables to a separable equation?

Here is the answer. In our case we consider a differential equation $y' = g(x)h(y)$. We assume g and h are continuous in a region D containing the initial point (a, b). Then there is always a solution (Peano), and if g' and h' are continuous, the solution is unique. Thus, the solution found using the method of separation of variables is unique if g' and h' satisfy these conditions.

The example $y' = y$ is instructive. (We solved this as a separable differential equation $y' = ky$ above with $k = 1$, $y \neq 0$.) Here $g(x) = 1$ and $h(y) = y$. Thus, the continuity and differentiability conditions are met because $g'(x) = 0$ and $h'(y) = 1$ imply that g, h, g', and h' are all continuous. And indeed there is a unique solution through any point (a, b). If the initial point is on the x-axis, however, the unique solution is $y = 0$, and this is not found by the method of separation of variables (one cannot divide by zero). Of course the fact that the method of separation of variables did not "find" the solution $y = 0$ has nothing to do with existence. The solution exists and is unique. It is only that the application of the separation of variables method was not valid in the case $y = 0$.

Whew! For most of us, this is all we need to know about the existence and uniqueness of solutions of a differential equation. However, we will now conclude our discussion with a few ideas that can point the way to further investigations of the subject at a subsequent point in your study of calculus.

To have uniqueness of the IVP $y' = F(x, y)$, $y(a) = b$, where $F(x, y)$ is continuous in a domain D, it is sufficient that $F(x, y)$ satisfy a Lipshitz condition in D, i.e. that there exist a constant M such that $|F(x, y_1) - F(x, y_2)| \leq M|y_1 - y_2|$. (Note: In a subsequent course on multivariable mathematics, you will learn that it is actually sufficient that $F(x, y)$ have continuous partial derivatives in the region D, and that the continuity of the partial derivatives in any closed bounded subregion of D implies a Lipschitz condition in that subregion.)

Example of non-uniqueness: The differential equation $y' = 3y^{\frac{2}{3}}$ has the "general solution" $y = (x+C)^3$. Just check that this is true by differentiation and substitution: $y = (x+C)^3$ implies $y' = 3(x+C)^2$, and $3y^{\frac{2}{3}} = 3((x + C)^3)^{\frac{2}{3}} = 3(x + C)^2$. But note that $y = 0$ is also a solution not of the general form. Of course if one specifies a domain D that does not touch the x-axis, then one has uniqueness within this domain. The function $3y^{\frac{2}{3}}$ has a continuous derivative $2y^{-\frac{1}{3}}$ when bounded away from the x-axis. For example $D = \{(x, y)|y > 0\}$ is such a domain.

0.5 Our Agenda for This Chapter

The prevalence of differential equations and their role in applications is that they are the mathematical language used to describe many biological and physical systems. We have seen that differential equations arise quite naturally in modeling systems that involve measurable quantities and their related rates of change as the system moves over time from one state to another. Hence, we cannot overemphasize the importance of differential equations in real-world applications. The world changes; differential equations describe how. Therefore, in this section we will be studying first approaches to modeling with and solving differential equations. This will involve applications, and both analytical and numerical techniques. Because we have already gotten started, we will take up where we left off by building on and extending the topics considered in this section. In particular, here is an agenda for the rest of the chapter as it grows out of our investigations so far.

1. Revisit the population model $\frac{dy}{dx} = ky$ and study further examples of growth ($k > 0$), and of decay ($k < 0$).
2. Revisit Separation of Variables and investigate why the method works.
3. Revisit slope fields and introduce Euler's Method.
4. Discuss analytical and numerical methods for studying more general population models.

Exercises: Problems **Check what you have learned!**
Videos: Tutorial Solutions **See problems worked out!**