Case Study: Torricelli's Law

Animation: Torricelli's Law **To get you going on the Case Study!**

We close section 2 with a *Case Study in Calculus* (CSC) that is an extended example of modeling rates of change. The purpose of a CSC is to consider a real application of calculus, with real data. The question we want to answer in the present section is straightforward enough:

Objective: To determine how long it would take a tank of given dimensions to empty its liquid contents through a bottom outlet hole.

In this CSC, we will play the role of a mathematician who works with a group of physicists. We will take note of certain physical *laws*, described to us by physicists, and use these laws to set up a differential equation that can be solved using methods that we have learned.

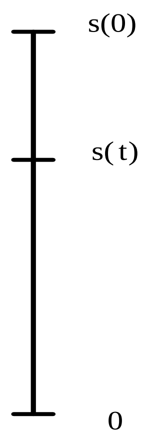
To a physicist a law is a statement for which there is solid empirical evidence of its truth. That is, the statement has never been known to be false. This test of truth, close to the legal standard of *beyond a shadow of a doubt*, is not good enough for mathematicians. Mathematicians require a valid deductive argument to *prove* the truth of a statement.

However, in the real world, compromises must be made. Our compromise as mathematicians will be to accept laws of physicists in order to set up our model, and then to apply mathematical analysis to derive an answer. Of course, we actually will have come full circle because interpreting the mathematical answer as it applies in a given situation may not be so crisp and clear-cut. The world of real data is messy, and approximations and simplifications have to be made. Keep these distinctions in mind as you go through the CSC. And by all means, have fun putting calculus to work.

Background: Falling Objects

Let m be the mass of a falling object, and let g be the acceleration due to gravity. Then $g \approx 9.8$ meters per second squared, or $g \approx 32.2$ feet per second squared.

Let $s(t)$, $s(t) \geq 0$ for all t, be the position of the object above the ground at time t according to the following coordinate axis:



If a is the acceleration and $v(t)$ is the velocity of the object at time t, then we have:

$$\begin{aligned} a &= -g \\ v &= -gt \\ s &= -\frac{gt^2}{2} + s_0 \end{aligned}$$

Thus, we can find the final velocity v_f at the final time t_f:

$$\begin{aligned} \frac{gt_f^2}{2} &= s_0 \\ t_f &= \sqrt{\frac{2}{g}s_0} \\ v_f &= -\sqrt{2gs_0} \end{aligned}$$

We can also write the Initial Value Problem that gives the velocity $v = \frac{ds}{dt}$ as a function of s:

$$\frac{ds}{dt} = -\sqrt{2gs}, \qquad s(0) = s_0$$

We have made the substitution $v(t) = \frac{ds}{dt}$, the rate of change of position with respect to time. The velocity is negative because the object is moving toward the origin on our positive scale. The differential equation involves the derivative of an unknown function $s(t)$. The equation $s(0) = s_0$ defines the *initial condition* and tells us that the starting position of the object is the greatest point on the coordinate line we have chosen. The differential equation and the initial condition is what we will need for the exploration of Torricelli's Law below.

The CSC: Torricelli's Law

Evangelista Torricelli (1608-1647) was an Italian physicist and mathematician who was a disciple of Galileo. He also served as Galileo's secretary, and is credited with discovering the following principle.

Torricelli's Law: Water in an open tank will flow out through a small hole in the bottom with the velocity it would acquire in falling freely from the water level to the hole.

This law is not at all obvious. Presumably, Torricelli discovered it through a combination of studying empirical data and demonstrating great physical insight. Here is one plausible explanation, although a complete derivation of the law requires a good understanding of hydrostatics. Imagine the water as a collection of tiny balls undergoing elastic collision. If we consider a vertical chain of balls, the kinetic energy of a falling ball will be completely transferred to the next one. Thus, the new initial velocity of the next ball equals the final velocity of the last, and so on down the line. When the last ball gets to the outlet hole, it will carry the same kinetic energy as if the top ball had fallen all the way down. But no matter how he arrived at it, Torricelli was indeed correct: today his law is a well-established scientific fact.

We can make Torricelli's Law more specific by introducing some notation. Suppose a cylindrical tank with cross-sectional area A has an outlet hole in the bottom. Further, suppose that $h(t)$ is the height of water above the outlet at time t, a is the area of the outlet hole, and $V(t)$ is the remaining fluid volume at time t.

Consider now the change in volume of water in the tank from time t to time $t + \Delta t$. This equals the amount of water that flows out through the outlet hole in time $[t, t + \Delta t]$. If we think of this water as filling a small cylindrical tube whose top is the outlet hole of cross-sectional area a, then the height of the tube is the velocity of a drop of water (if it were constant) times the time. By Torricelli, the initial velocity of a drop is $\sqrt{2gh}$ (the final velocity from the background analysis of falling objects above). Thus, the volume of the tube in the interval $[t, t + \Delta t]$ is approximately $a\sqrt{2gh}\,\Delta t$. Equating the change of volume of the tank with the volume of water in this tube and using this approximation for the latter, we get

$$\begin{aligned} V(t) - V(t + \Delta t) &\approx a\sqrt{2gh}\,\Delta t \\ \frac{V(t) - V(t + \Delta t)}{\Delta t} &\approx a\sqrt{2gh} \\ \frac{V(t + \Delta t) - V(t)}{\Delta t} &\approx -a\sqrt{2gh} \\ \lim_{\Delta t \to 0} \frac{V(t + \Delta t) - V(t)}{\Delta t} &= -a\sqrt{2gh} \\ \frac{dV}{dt} &= -a\sqrt{2gh} \end{aligned}$$

Now, we also have that the volume V of remaining water at time t equals the cross-sectional area A of the tank times the height h of water at time t. Thus, the rate of change of volume with respect to time is the cross-sectional area of the tank times the rate of change of the height of water with respect to time. Putting these comments together with the above equation for $\frac{dV}{dt}$ will give a differential equation involving $\frac{dh}{dt}$ and h, but we will leave that for the setup part of the CSC.

Thus far, we have collected information relevant to answering the question posed at the beginning. Let's restate the objective a bit more precisely, and then make an inventory of that information.

The Objective of the CSC: To determine, from the background information above, how long it would take a cylindrical tank of given dimensions to empty through a bottom outlet hole of known diameter.

A possible scenario for needing to solve this problem is that you are a consultant for an oil company. The company wants to know how long it will take to empty its oil storage tank into its tanker trucks.

Nature of the information that is available: Dimensions of a tank, Torricelli's Law, Equation of Motion of a falling object.

All of this information appears above. We have to make sense of it. To do so, we will bring to bear our analytical skills, and complete a number of steps using some of the mathematical tools we have learned. Here are the steps.

Steps to follow to analyze the information: First, model the physical situation as a differential equation. Next, solve the differential equation. And finally, relate the solution to the question posed in the objective.

It is always a good idea to list the tools that we will use: derivatives, antiderivatives, our brains. The CSC is an extended application of calculus. It is *extended* in the sense that we will have to complete several stages to arrive at a solution. So, as always, thinking is important. In fact, the less routine a problem, the more important thinking becomes. But don't worry. A major purpose of the CSC is to learn to think clearly about such problems. Toward that end, we have structured the steps of the analysis in the form of a report that you will complete. The main sections of the report are as follows (see homework).

Setup: This is what is called the *modeling phase*. The model is often a differential equation. In this section, we derive the equation to be solved, being careful to be clear about the reasoning– scientific or mathematical. We do not solve the differential equation here, but in the next section.

Thinking and Exploring: Now it is time to think about the mathematics. We will suggest some activities for you to carry out to develop an understanding of the differential equation and its relationship to the solution of the barrel problem stated in the objective above.

Applet: Function Grapher **Try it!**

Interpretation and Summary: Now that you have modeled the barrel problem, and thought about the model and derived mathematical facts, it is time to interpret and summarize the mathematical results in terms of the original objective. This is a very important part of the report because mathematicians often have to explain their work to non-mathematicians, and be convincing about the proposed course of action. Pretend that your synopsis is going to appear in the next issue of a magazine such as *Scientific American*. Include enough details so that a reader would learn what the major issues of the report are, and how you went about addressing them. What will you want to tell readers about your success with regard to the original stated objective of the investigation? You should take care to write in complete sentences using correct rules of standard English grammar. Make the report interesting, compelling. Ask yourself: Would a friend enjoy reading it? Would an oil company executive follow your advice? Both answers should be yes.

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**