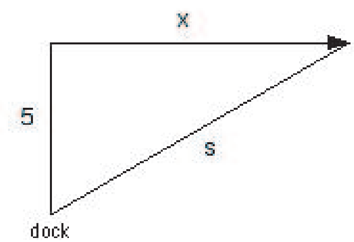
Related Rates

One of the applications of mathematical modeling with calculus involves related-rates word problems. Suppose we have an equation that involves two or more quantities that are changing as functions of time. Then differentiating the equation implicitly with respect to time gives an equation that involves the rates of change of these quantities. By relating the rates in this way, we often can answer interesting questions about the model that we use to specify the original problem.

Example 1: Suppose a ship is traveling at a speed of 28 knots (nautical miles per hour) in a line perpendicular to a dock on the shore as in the sketch below. How fast is the distance between it and the dock increasing 15 minutes after it passes directly opposite the dock five miles away?



Consider the sketch that describes the situation. Here x is the distance traveled by the ship; thus, we are *given* that

$$\frac{dx}{dt} = 28 \text{ nautical miles per hour}$$

and we are asked to *find*

$$\frac{ds}{dt} \text{ when } t = \tfrac{1}{4} \text{ hour.}$$

We begin by writing an equation that involves x and s:

$$s^2 = x^2 + 25$$

Then we differentiate with respect to t:

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

Hence, solving for $\frac{ds}{dt}$ we get:

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

Now, we can *answer the question*: When $t = 1/4$, $x = (28)(1/4) = 7$. Thus, from the Pythagorean Theorem, $s = \sqrt{25+49} = \sqrt{74}$. Hence,

$$\frac{ds}{dt} = \frac{7}{\sqrt{74}}(28) \approx 22.78 \text{ knots}$$

Example 2: How fast is the area of a square changing when the side is of length 10 cm and is increasing at a rate of 4 cm/s? To answer the question we begin with $A = s^2$ where s is the length of the side and A is the area. Differentiating with respect to t we get:

$$\frac{dA}{dt} = 2s\frac{ds}{dt}$$

Thus, when $s = 10$, $\frac{dA}{dt} = (2)(10(4) = 80$ square centimeters per second.

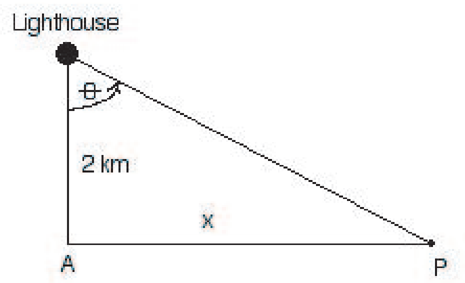
These examples illustrate a general procedure for solving Related Rates problems:

1. Begin by making a sketch whenever you can.

2. Define symbols and write down what is given.

3. Write an equation that links the variables.
4. Differentiate the equation implicitly with respect to time to get an equation that links (that is, relates) the rates.
5. Answer the question by substituting in specific values that are given.

Example 3: A lighthouse is located on a mound of rock out in the ocean 2 km from the nearest point A on a straight shoreline. If the lamp rotates at a rate of 3 revolutions per minute, how fast is the lighted spot P on the shoreline moving along the shoreline when it is 4 km from point A? We start by making a sketch.



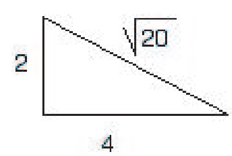
With reference to the sketch, we have:

$$\tan\theta = \frac{x}{2}$$
$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$$
$$\frac{dx}{dt} = 2\sec^2\theta\,\frac{d\theta}{dt}$$

Now, with reference to the given information, we have to change *revolutions* to *radians* so that we can use the differentiation formulas above: 1 revolution per minute equals 2π radians per minute. Thus,

$$\frac{d\theta}{dt} = 6\pi \text{radians per minute}$$

And from the triangle below, we see that when $x = 4$, $\sec\theta = \sqrt{20}/2$.



Therefore, we can answer the question: when $x = 4$, we have

$$\frac{dx}{dt} = 2\left(\frac{\sqrt{20}}{2}\right)^2 6 = 60\pi \;\; \text{km/min}$$

Exercises: Problems **Check what you have learned!**
Videos: Tutorial Solutions **See problems worked out!**