Velocity and Acceleration

In a previous section we defined the notions of *average velocity* and *instantaneous velocity*. For an object traveling on a straight line, whose position at time t is given by the function $x = x(t)$,

$$\text{Average velocity on the interval } [t, t+h] \; = \frac{x(t+h) - x(t)}{h}, \text{ and}$$

$$\text{Instantaneous velocity } v(t) \text{ at time } t \; = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}.$$

Recall also from our previous discussion that the acceleration $a(t)$ of the object at time t is defined similarly from the velocity function:

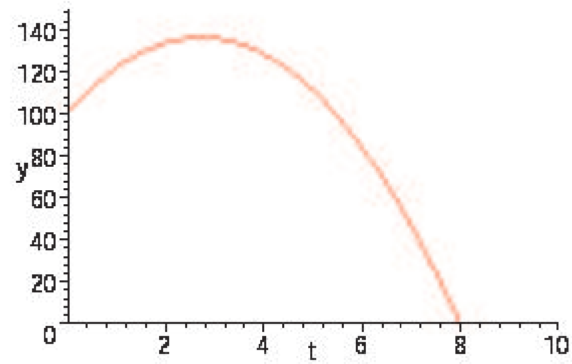
$$a(t) = v'(t) = x''(t).$$

Example 1: An object dropped from a cliff has acceleration $a = -9.8$ m/sec^2 under the influence of gravity. Let its height at time t be given by $s(t)$. Then its motion is described by the initial-value problem

$$\frac{d^2 s}{dt^2} = -9.8, \quad s(0) = s_0, \quad s'(0) = v_0$$

where s_0 is the height of the cliff and v_0 is the object's initial velocity. We solved an initial-value problem similar to this one in the previous section. Integrating both sides of the differential equation we get $s'(t) = -9.8t + C_0$, and a second integration gives $s(t) = -4.9t^2 + C_0 t + C_1$, where C_0 and C_1 are arbitrary constants. The initial conditions enable us to evaluate the constants. From $s'(0) = v_0$ we obtain $-9/8 \cdot 0 + C_0 = v_0$, and hence $C_0 = v_0$. And from $s(0) = s_0$ we then obtain $-4.9 \cdot 0^2 + v_0 \cdot 0 + C_1 = s_0$, and hence $C_1 = s_0$. The particular solution that describes the motion of the object is thus $s = -4.9t^2 + v_0 t + s_0$.

Example 2: Let us apply the result of Example 1 to a particular case. Suppose that a baseball is thrown upward from the roof of a 100 meter high building. It hits the street below eight seconds later. What was the initial velocity of the baseball, and how high did it rise above the street before beginning its descent? Letting $y(t)$ be the height of the baseball above the street, and assuming that it was thrown at time $t = 0$, we know from Example 1 that $y(t) = -4.9t^2 + v_0 t + 100$. When $t = 8$ the baseball hits the street,



i.e. $s(8) = 0$. Thus from $0 = -4.9 \cdot 8^2 + v_0 \cdot 8 + 100$ we find that the initial velocity was 26.7 meters/sec^2. Finally, the highest point in the baseball's trajectory, the point when $s(t)$ achieves its maximum value, is when $v(t) = s'(t) = -9.8t + v_0 = 0$. This occurs when $t_{\text{max}} = v_0/9.8 = 26.7/9.8 = 2.72$ seconds (approximately). At this time the baseball is

$$s(t_{\text{max}}) = -4.9(t_{\text{max}})^2 + v_0(t_{\text{max}}) + 100 = 136.37$$

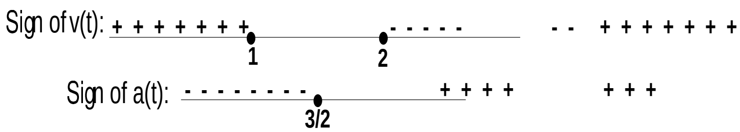
meters above the street.

Example 3: Suppose an object moves on the x-axis so that its position at time t is given by $x(t) = 2t^3 - 9t^2 + 12t + 6$, $-\infty < t < \infty$. When is the object at rest? When is it moving to the right? When is it speeding up? When is it slowing down? These questions are answered by studying $v(t) = x'(t)$ and $a(t) = x''(t)$:

$$v(t) = x'(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2)$$

$$a(t) = x''(t) = 12t - 18 = 12\left(t - \frac{3}{2}\right)$$

The object is at rest when $v(t) = 0$. This occurs when $t = 1$ and when $t = 2$. The position of the object at these times is $x(1) = 11$ and $x(2) = 10$. The object is moving to the right when $v(t) > 0$ and to the left when $v(t) < 0$. A sign graph for $v(t)$, below,



shows that the object is moving right when $t \in (-\infty, 1) \cup (2, \infty)$, and it is moving left in the interval $(1, 2)$. We must be careful when considering when it is speeding up or slowing down. The *velocity* $v(t)$ can be positive or negative; the *speed* of the object is $|v(t)|$. If the object is moving to the right it is speeding up when $a(t) > 0$, but if the object is moving left it is speeding up when $a(t) < 0$ (i.e. the velocity is negative and is becoming more negative). More simply, it is speeding up when $v(t)$ and $a(t)$ have the same sign and slowing down when they have opposite signs. From the sign graphs, above, we can easily compare the signs and see that the object is speeding up on $(1, \frac{3}{2}) \cup (2, \infty)$, and slowing down on $(-\infty, 1) \cup (\frac{3}{2}, 2)$.

Example 4: An object is traveling along the x-axis and its position at time t is given by $x = f(t) = 3\sin\frac{\pi}{2}t$ meters to the right of the origin. At time $t = 1$ describe its position, velocity and acceleration. These three quantities are just the values of $f(1)$, $f'(1)$ and $f''(1)$. Since $x' = f'(t) = 3\frac{\pi}{2}\cos\frac{\pi}{2}t$ and $x'' = f''(t) = -3\left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2}t$, we can state that the position of the object is $3\sin\left(\frac{\pi}{2} \cdot 1\right) = 3$; i.e. it is 3 meters to the right of the origin. Its velocity is $3\frac{\pi}{2}\cos\left(\frac{\pi}{2} \cdot 1\right) = 0$ meters/second; i.e. it is momentarily at rest. And its acceleration is $-3\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2} \cdot 1\right) = -3\left(\frac{\pi}{2}\right)^2 = -7.4$ meters/second^2; i.e. it is "speeding up toward the left".

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**