Antiderivatives and Initial Value Problems

We have described the problem of differentiation as "half of calculus". The problem of finding the slope of a curve led to a general definition of derivative and was followed by development of general techniques for finding derivatives of given functions. This was the substance of *differential calculus*. Interpretations of the derivative as *rate of change*, *slope of a graph*, and *velocity of a moving object* point to a great variety of applications that characterize differential calculus as the science of dynamic behavior.

Humanity is prone to walk backwards. We like to run videos in reverse. We like to undo our mistakes. And for every procedure known to humans we want to know what happens if we reverse the procedure.

So far we have operated differentiation only in the forward direction. But now we ask the reverse question—if we are given the derivative of a function f can we find an *antiderivative*; i.e. can we find a function F such that $F'(x) = f(x)$? And what interpretations and applications would follow from reversing the process of differentiation?

Definition 1: An *antiderivative* of a function f on an interval I is another function F such that $F'(x) = f(x)$ for all $x \in I$.

Example 1: Find an antiderivative of $f(x) = 2x$. From our knowledge of differentiation techniques we immediately think of x^2. Can we find others? Yes. Indeed we can write down many more: $x^2 + 1$, $x^2 - 3$, $x^2 + 72$. And in fact $f(x) = x^2 + C$ is an antiderivative of f for any constant C. This is the end of our search, for we will show that any two antiderivatives differ only by a constant.

If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ on an interval I, then

$$\frac{d}{dx}(G(x) - F(x)) = G'(x) - F'(x) = f(x) - f(x) = 0$$

for every $x \in I$. Thus $G-F$ is a differentiable function whose derivative is identically zero on I. The following theorem shows that any such function must be a constant C; i.e. $G(x) - F(x) = C$, or $G(x) = F(x) + C$ for all $x \in I$.

Theorem 1: Suppose that h is differentiable in an interval I and $h'(x) = 0$ for all $x \in I$. Then h is a constant function; i.e. $h(x) = C$ for all $x \in I$, where C is a constant.

The proof is an immediate consequence of the mean value theorem. For if a and b are any two points in I, there is a point $c \in I$ where $h(b) - h(a) = (b - a)h'(c)$. But $h'(c) = 0$, so $h(b) = h(a)$.

This settles the antiderivative problem. If $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative must be of the form $F(x) + C$, where C is a constant. We refer to $F(x) + C$ as the *general antiderivative* and denote it by

$$\int f(x)dx$$

which is called the *indefinite integral of f*. Note that the indefinite integral is nothing but the general antiderivative of f, i.e.

$$\int f(x)dx = F(x) + C$$

where F is one antiderivative of f, and C is an arbitrary constant.

Example 2: In the language of indefinite integrals, the result of Example 1 is just the statement

$$\int 2xdx = x^2 + C.$$

Example 3: Each of our differentiation formulas has a companion *integral* formula. For example

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int e^x\,dx = e^x + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Each of these formulas is verified by simply differentiating the right hand side.

Note too that just one step removed from these basic integrals are integrals such as

$$\int \cos 3x\,dx = \frac{1}{3}\sin 3x + C$$
$$\int \sin 5x\,dx = -\frac{1}{5}\cos 5x + C$$
$$\int e^{7x}\,dx = \frac{1}{7}e^{7x} + C$$

Here the formulas amount to undoing instances of the chain rule involving constants, and can be verified as usual by differentiating the right hand sides.

The following theorem gives a useful property of indefinite integrals. Just as for derivatives, the indefinite integral of a sum of functions is the sum of the indefinite integrals of the terms.

Theorem 2: Suppose the functions f and g both have antiderivatives on the interval I. Then for any constant a, the function $af + g$ has an antiderivative on I and

$$\int (af+g)(x)\,dx = a\int f(x)\,dx + \int g(x)\,dx$$

The proof follows from the fact that if F is an antiderivative of f (i.e., $F' = f$), and G is an antiderivative of g (i.e., $G' = g$), then $aF + G$ is an antiderivative of $af + g$. We verify this by differentiation: $(aF+G)' = aF' + G' = af + g$.

Example 4: Theorem 2 allows us to find the indefinite integral of any sum of functions whose indefinite integrals we already know, for instance those from the list of Example 3. A typical example is:

$$\int (6x^5 + 4\cos(x) - \frac{1}{x})\,dx = 6\frac{x^6}{6} + 4\sin(x) - \ln|x| + C$$

That the answer is correct can be verified by differentiating the right hand side, thereby obtaining the function under the integral sign.

Differential Equations: Finding an antiderivative of f can be thought of as solving the equation $\frac{dy}{dx} = f(x)$ for the unknown function y. Such equations that involve one or more derivatives of an unknown function are called *differential equations*. They are of fundamental importance in mathematical modeling. Solving a differential equation means finding a function $f(x)$ that satisfies the equation identically when substituted for the unknown function y.

Example 5: Solve the differential equation $y' = 2x + \sin x$. This is just the antiderivative problem, thus the *general solution* is $y = x^2 - \cos x + C$. Checking the solution means substituting it for y in the differential equation. We see that it does indeed satisfy the equation:

$$\begin{aligned}\frac{d}{dx}(x^2 - \cos x + C) &= \frac{d}{dx}(x^2) - \frac{d}{dx}(\cos x) + \frac{d}{dx}(C) \\ &= 2x - (-\sin x) + 0 \\ &= 2x + \sin x.\end{aligned}$$

Example 6: Solve the *second-order* differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

The equation is called *second-order* because it involves the second derivative of the unknown function. We do not yet have techniques for solving this equation, but we can easily verify that it has many solutions. For example $\sin x$ is a solution as can be seen by substituting into the equation. [If $y = \sin x$, then $y' = \cos x$ and $y'' = -\sin x$. Hence $y'' + y = 0$.] Other solutions are: $\cos x$, $3\sin x$, $-5\cos x$, $2\sin x - 3\cos x$. Indeed it turns out that the general solution of this differential equation is $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are independent arbitrary constants. We expect that the number of arbitrary constants in the general solution of a differential equation is the *order* of the equation, i.e. the highest order of a derivative that appears in the equation.

In most applications of differential equations the problem at hand will provide additional conditions that enable us to determine values for the arbitrary constants. Then we seek a *particular solution* that satisfies not only the differential equation but also the additional conditions.

Definition 2: An *initial-value problem* is a differential equation together with enough additional conditions to specify the constants of integration that appear in the general solution. The *particular solution* of the problem is then a function that satisfies both the differential equation and also the additional conditions.

The term *initial-value problem* comes from the fact that in many applications of differential equations the independent variable is time t, and the additional conditions specify the state of the system at some initial time, say $t = 0$.

Example 7: Solve the initial value problem $\frac{dx}{dt} = 2t + \sin t$ subject to the initial condition $x(0) = 0$. This problem might be modelling, for example, the motion of an object moving on the x-axis, with velocity $\frac{dx}{dt}$ at time t given by the differential equation. Initially, at time $t = 0$, the object was at the origin. We have already obtained the general solution $x(t) = t^2 - \cos t + C$ of this equation in Example 5. (We have only changed the names of the independent and dependent variables.) The initial conditon enables us to determine a particular value of the constant C. Substituting $t = 0$ into the general solution we obtain

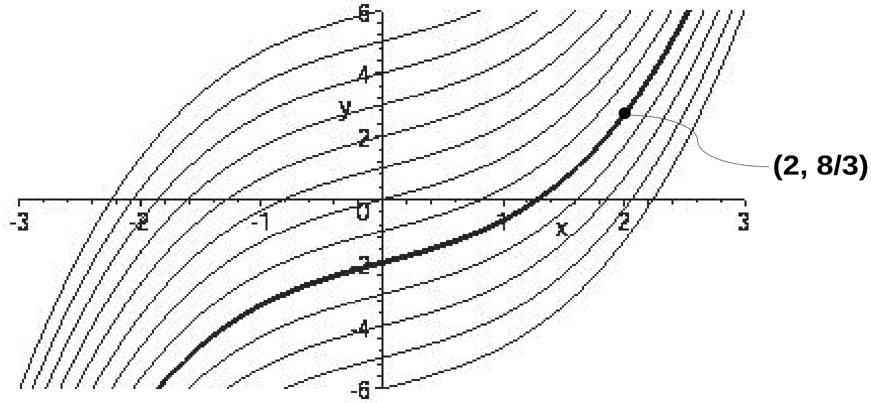
$$0 = 0^2 - \cos 0 + C = 0 - 1 + C = -1 + C.$$

Thus we must have $C = 1$ in order to satisfy the initial condition. And we then obtain the *particular solution* $x = t^2 - \cos t + 1$. Knowing the position of the object at the initial time $t = 0$ and its velocity at any time t, the solution gives us the position of the object at all future times.

Example 8: Solve the differential equation $y' = x^2 + 1$ subject to the additional condition $y(2) = 8/3$. This is again just an antiderivative problem. We solve the equation by "integrating" both sides, thus:

$$\int \frac{dy}{dx}\, dx = \int (x^2 + 1)dx$$

The integral on the left-hand side is y since the indefinite integral is just antidifferentiation. And the integral on the right is $(1/3)x^3 + x + C$. The general solution is thus $y = (1/3)x^3 + x + C$. (It is only necessary to add the arbitrary constant on one side since otherwise we can combine them into a single constant.) Applying the initial condition we must have $8/3 = (1/3)2^3 + 2 + C$, and this yields $C = -2$. The desired particular solution is $y = (1/3)x^3 + x - 2$.



Notice that the general solution is a family of curves that differ only by a vertical translation. The plot shows members of the family for values of C ranging from -6 to 6. The geometric significance of the initial condition $y(2) = 8/3$ is apparent—it "picks out" from the family of curves the particular member of the family that passes through the point $(2, 8/3)$. This is the member corresponding to $C = -2$.

Example 9: Solve the initial-value problem

$$y'' = \cos x, \quad y'\left(\frac{\pi}{2}\right) = 2, \;\; y\left(\frac{\pi}{2}\right) = \pi.$$

This time the differential equation is of order two, and two initial conditions are given. Initial-value problems that specify the values of a function and its derivatives at a single point are very common. For a second-order equation, for example, this often comes about by specifying the initial position and velocity (momentum) of an object. For the example at hand we solve the problem by performing two integrations. Integrating both sides of the equation we have

$$\int y'' dx = \int \cos x \, dx,$$

or $y' = \sin x + C_1$, where C_1 is an arbitrary constant. Integrating again

$$\int y' dx = \int (\sin x + C_1) dx$$

we obtain the general solution $y = -\cos x + C_1 x + C_2$, where C_2 is a second arbitrary constant. Finally, we obtain the desired particular solution by applying the initial conditions: setting $x = \frac{\pi}{2}$ and the values of y and y' to π and 2 respectively:

$$2 = \sin\frac{\pi}{2} + C_1$$

$$\pi = -\cos\frac{\pi}{2} + C_1\left(\frac{\pi}{2}\right) + C_2$$

Solving these two equations for the constants C_1 and C_2 we find that $C_1 = 1$ and $C_2 = \pi/2$. Thus, finally, $y = -\cos x + x + \pi/2$.

Summary: Differential equations are at the heart of modelling motion in dynamic systems. They provide the language in which we can describe the state of a physical system. For example an object in motion may be described in terms of its position, velocity, and acceleration. An equation relating these properties is thus an equation involving a function and its first and second derivatives. Initial-value problems have additional conditions that allow a *particular solution* to be picked out from the *general solution*. Not surprisingly we will see that differential equations occupy a great deal of our attention from now on.

Exercises: Problems **Check what you have learned!**

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