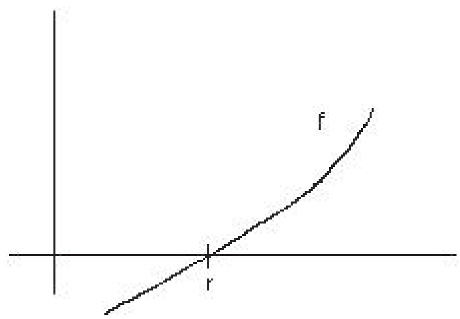
Newton's Method

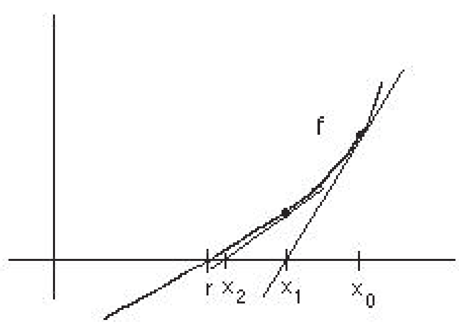
The tangent line at a point on the graph of a differentiable function can give a great deal of information about the function. It is this fact that makes the concept of the derivative so powerful, and has maintained its importance since its introduction in the seventeenth century.

For example, suppose we want to find a *root* of the equation $f(x) = 0$; that is, a number r such that $f(r) = 0$. The number r is called a *zero* of f.



We will describe a procedure, called *Newton's Method*, to find the root using tangent lines. The method is very beautiful in that it is easy to explain, and works very well in many circumstances. The idea is to use points where certain tangent lines intersect the x-axis to get close to the root.

To be specific, assume that f is differentiable. Choose a starting value x_0 near r on the x-axis. Then the tangent line at $(x_0, f(x_0))$ in many cases will intersect the x-axis in a point x_1 closer to r. Next, we repeat what we just did. That is, we draw a new tangent line at $(x_1, f(x_1))$ and hope that the point x_2 where it intersects the x-axis will be even closer to r. It most often is. Thus, we continue to repeat, defining a sequence $x_0, x_1, x_2, x_3, \ldots, x_n, \ldots$ such that $x_n \to r$.



To implement the procedure, we need an expression for x_n. Note that an equation of the tangent line at $(x_0, f(x_0))$ is $y = f(x_0) + f'(x_0)(x - x_0)$. So, with $y = 0$, we find x_1 by solving the equation:

$$\begin{aligned} 0 &= f(x_0) + f'(x_0)(x_1 - x_0) \\ f'(x_0)x_1 &= -f(x_0) + x_0 f'(x_0) \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

Thus, we have found a formula for x_1 in terms of x_0 and functions of x_0.

But there is nothing special here about x_0 and x_1. Given x_n, we can determine x_{n+1} in a similar way. That is,

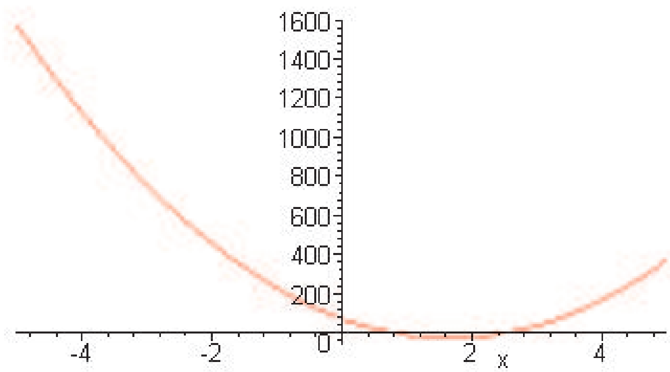
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad\qquad n = 0, 1, 2, \ldots$$

The above procedure with this formula is known as *Newton's Method*.

Example 1: Use Newton's Method to find the zeros of $f(x) = 12(3x^2 - 10x + 6)$. We calculate the derivative: $f'(x) = 12(6x - 10)$. Thus for $n \geq 0$,

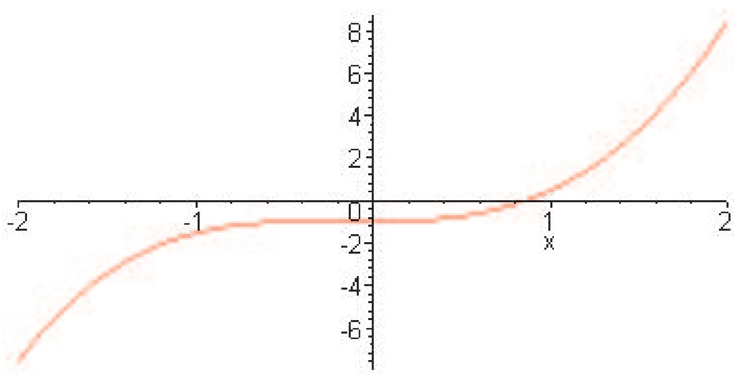
$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{12(3x_n^2 - 10x_n + 6)}{12(6x_n - 10)} \end{aligned}$$

Below is a sketch of the graph.



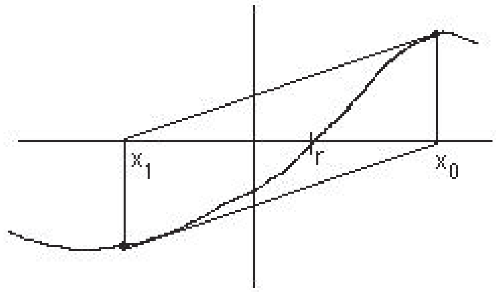
The equation has two roots. If we let $x_0 = 3$, then using a computer or programmable calculator we get approximately $x_1 = 2.625000000$, $x_2 = 2.551630435$, $x_3 = 2.548589014$, $x_4 = 2.548583771$. In fact, $x_{10} = 2.548583770$. If, on the other hand, we let $x_0 = 0.5$, then Newton's method will give an approximation to the other root: $x_1 = .7500000000$, $x_2 = .7840909091$, $x_3 = .7847493172$, $x_4 = .7847495630$; $x_{10} = .7847495630$.

Example 2: To solve the equation $x^3 = \cos x$, we let $f(x) = x^3 - \cos x$. Then $f'(x) = 3x^2 + \sin x$. Here is a sketch of the function that we can use to define a starting value x_0.



If we let $x_0 = 0.5$, we get approximately $x_1 = 1.112141637$, $x_2 = .9096726937$, $x_3 = .8672638182$, $x_4 = .8654771353$, $x_5 = .8654740331$. Also, $x_{10} = .8654740331$.

Newton's Method can fail as seen, for example, in the following sketch. However, it is a surprisingly powerful technique for the simplicity of the idea.



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