Derivatives of Exponential and Logarithm Functions

We began our study of calculus with a review of polynomials and trigonometric functions. Now, we have added the exponential and logarithmic functions to the list of functions that we are calling *elementary functions*. Their properties also should be familiar to us now. The elementary functions are all essential because they are very important in modeling real-world situations.

For purposes of modeling, in addition to the functions themselves, we also need to be familiar with their derivatives. As we have seen, it is relatively straightforward to develop a formula for the derivative of a polynomial using algebra. The derivative of the sine was a bit more complicated because we had to analyze and compute a particular limit. Once done, the derivatives of the other trig functions followed rather readily. The case of the exponential function is similar. Once we have computed a particular limit, the derivatives of the exponential and logarithmic functions will follow in a straightforward mannner by calculating the limits of the difference quotients directly from the definition of the derivative.

The Derivative of $y = e^x$: Let $y = e^x$. Then

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= e^x\left(\lim_{h\to 0}\frac{e^h-1}{h}\right)\end{aligned}$$

Therefore, the derivative will exist if the limit $\lim_{h\to 0}\frac{e^h-1}{h}$ exists. But we have seen this limit before in two different contexts. In Section 2.1.4 we interpreted it to be the limit of the difference quotients of $y = e^x$ at the point $x = 0$. We now know that in fact this is precisely the definition of the derivative of $y = e^x$ at $x = 0$, which furthermore is the slope of the tangent line to the graph of the function there. From a table of values, we estimated this limit to be equal to 1. We now can return to the earlier context in Section 1.5 in which we defined the number e to be the base of that exponential function whose tangent line to its graph at the point $(0, 1)$ is of slope 1. When we did that, we were not able to make precise the notion of the tangent line at a point on an arbitrary curve. The derivative allows us to overcome that obstacle. The value of the limit $\lim_{h\to 0}\frac{e^h-1}{h}$ is the derivative of the exponential function $y = e^x$ at $x = 0$, which equals the slope of the tangent line at $(0, 1)$ of the graph, or 1. Thus, we now can complete the calculation of the derivative of $y = e^x$ for any x:

$$\frac{dy}{dx} = e^x\left(\lim_{h\to 0}\frac{e^h-1}{h}\right) = e^x\cdot 1 = e^x$$

Applet: Limits of Functions **Try it!**

We now give as a theorem the chain rule form of this result.

Theorem 1: Let u be a function of x. Then

$$\tfrac{d}{dx}e^u = e^u\tfrac{du}{dx}$$

In words, the theorem tells us that the derivative of an exponential function is produced by multiplying the function by the derivative of the exponent.

Example 1: If $y = e^{17x}$, then using the theorem we get $\frac{dy}{dx} = 17e^{17x}$.

Example 2: Let $y = e^{\sin x}$. Then $y' = (\cos x)e^{\sin x}$.

The Derivative of $y = \ln x$: Now that we know the derivative of the exponential function, we can find the derivative of its inverse $y = \ln x$ by implicit differentiation:

$$\begin{aligned} y = \ln x \quad &\Leftrightarrow \quad e^y = x \\ e^y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{e^y} \\ &= \frac{1}{x} \end{aligned}$$

where in the last equality we use the fact that $e^y = x$.

So, we have found a function whose derivative is $1/x$, namely, $\ln x$. But we have to be somewhat careful here. The natural log is only defined for positive values of x. Suppose x is negative. Then what?

If $x < 0$, then $|x| = -x$ and $\frac{d}{dx}\ln(-x) = \frac{1}{-x} \cdot -1$ by the chain rule. Thus, $\frac{d}{dx}\ln(-x) = \frac{1}{x}$. Hence, we have shown that the following theorem is true.

Theorem 2: Let u be a function of x. Then

$$\tfrac{d}{dx}\ln|u| = \tfrac{1}{u}\tfrac{du}{dx}$$

Example 3: By the chain rule, the derivative of $y = \ln 7x$ is $y' = \frac{1}{7x} \cdot 7 = \frac{1}{x}$.

Example 4: If $y = \ln x^2$, then $y' = \frac{1}{x^2}(2x) = \frac{2}{x}$. We get the same answer if we use a property of logarithms to simplify first: $y = \ln x^2 = 2\ln x$; so, $y' = 2\frac{1}{x}$.

The Calculus Standards: e^x and $\ln x$

In calculus we rely on $y = e^x$ as the standard exponential function, and on $y = \ln x$ as the standard logarithm. They have the simplest differentiation formulas. Moreover, as we have seen, we can obtain any other general exponential or logarithm as follows, $a > 0$:

$$a^x = e^{x\ln a}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Example 5: Let $y = 2^x$. Then to find its derivative we rewrite the function as $y = e^{x\ln 2}$ and use the chain rule: $y' = e^{x\ln 2}\ln 2 = \ln 2\ 2^x$. Note that $\ln 2$ is just a constant.

Example 6: Even though $y = x^x$ does not have a constant base, we proceed to rewrite it as $y = x^x = e^{\ln x^x} = e^{x\ \ln x}$. Then, using the chain rule, we get $y' = e^{x\ \ln x}\left(\ln x + x\ \frac{1}{x}\right)$. Hence, $y' = x^x\left(\ln x + 1\right)$.

Example 7: We can also find the derivative of $y = x^x$ using a method called *Logarithmic Differentiation*. In this method we take logs of both sides and differentiate applying the chain rule. First, taking logs we get $\ln y = \ln x^x = x\ \ln x$. Then we get $\frac{1}{y}\ y' = \ln x + x\ \frac{1}{x}$. Hence, $y' = y\ \left(\ln x + 1\right) = x^x\left(\ln x + 1\right)$, the same answer as above.

The Equation $y' = ky$

Suppose y is a function of x and satisfies the equation $y' = ky$ where k is a constant. Hence, the equation tells us that y is a constant multiple of of its own derivative. If $k = 1$, then $y = e^x$ has this property and thus solves the equation. In fact, the chain rule leads us to conclude that $y = e^{kx}$ solves the equation for any k. That is, the derivative of $y = e^{kx}$ is $ke^{kx} = ky$.

The equation $y' = ky$, because it includes a derivative, is an example of a *differential equation*. Differential equations are very important in calculus especially because of their centrality in modeling real-world phenomena. In fact, this particular differential equation is very important in the study of many populations, a topic that we will take up later.

The method we have used to solve the equation is called *guess-and-check*: we guessed a solution and then showed that it satisfied the equation. We have not shown that this is the *only* solution. Can you think of others? We will return to the question of uniqueness of solutions when we continue the study of differential equations shortly.

Exercises: Problems **Check what you have learned!**

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