Implicit Differentiation
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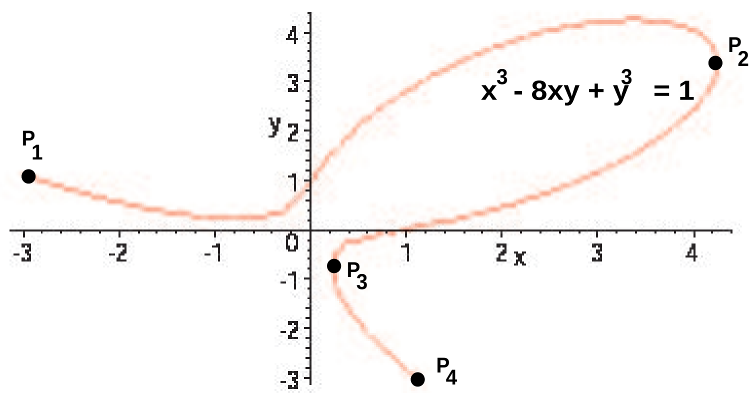
Derivatives are a powerful tool for studying curves in the xy-plane. For any curve that is the graph of a function $y = f(x)$ the derivative y' gives us information about slope, maxima and minima, and general tendencies such as *increasing* or *decreasing*.

Many curves, however, are not the graphs of functions. A circle of radius 1, for example, does not pass the "vertical line test" and hence is not the graph of a function. It is, however, the graph of the *equation* $x^2 + y^2 = 1$. And we have seen that this equation defines *two* functions $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ whose graphs are the upper and lower semi-circles that make up the circle. The circle can thus be broken into two pieces each of which can be studied using derivatives. In this example we were able to solve the given equation for y. But we will not always be so lucky.

The equation $x^3 - 8xy + y^3 = 1$, for example, resists our most clever efforts to explicitly solve for y as a function of x. Indeed, as we can see from the graph of the equation plotted below, the curve can be broken into three pieces each of which passes the vertical line test (the piece from P_1 to P_2, the piece from P_2 to P_3, and the piece from P_3 to P_4). It would seem, therefore, that the given equation determines *implicitly* three functions of x; let us call them $y = f_1(x)$, $y = f_2(x)$, and $y = f_3(x)$. We do not have *explicit* formulas for these three functions, however, and thus our techniques developed so far are of little help. We will see how to overcome this difficulty using a very important technique called *implicit differentiation*.



The general setting for our discussion of implicitly defined functions is an equation $F(x, y) = 0$, where F is an expression containing the two variables x and y. The graph of the equation is, in general, a curve, but as we have seen it is not usually the graph of a function since it does not satisfy the vertical line test. A function $f(x)$ is said to be *implicitly defined* by the equation if $F(x, f(x)) = 0$ on some interval I. Our wish is to find the derivative of $f(x)$ without explicitly solving the equation (which we normally will not be able to do).

Example 1: The functions $\sqrt{1-x^2}$ and $-\sqrt{1-x^2}$ are implicitly defined by the equation $x^2 + y^2 = 1$. For example, substituting the first function into the equation we have

$$x^2 + \left(\sqrt{1-x^2}\right)^2 = x^2 + (1-x^2) = 1.$$

Thus the function is a "solution" of the equation.

The chain rule is the key to finding the derivative of an implicitly defined function. We will illustrate this in examples:

Example 2: Consider one of the functions $f(x)$ defined implicitly by the equation $x^2 + y^2 = 1$. Then $f(x)$ satisfies the equation $x^2 + (f(x))^2 = 1$. Differentiating the equation, and using the chain rule, we have

$$2x + 2f(x)f'(x) = 0,$$

and solving the resulting equation for $f'(x)$ we have

$$f'(x) = -\frac{x}{f(x)}.$$

A formula for the derivative has thus been obtained, albeit in terms of the function $f(x)$ itself. This is natural in light of the fact that there is more than one function defined implicitly by the equation $x^2+y^2=1$. If $f(x)=\sqrt{1-x^2}$ our formula for the derivative gives $f'(x)=-x/f(x)=-x/\sqrt{1-x^2}$. And if $f(x)=-\sqrt{1-x^2}$ the formula gives $f'(x)=-x/f(x)=-x/\left(-\sqrt{1-x^2}\right)=x/\sqrt{1-x^2}$. Notice that these results agree with our previous methods for computing these derivatives.

Example 3: We repeat Example 2, simplifying our notation. Given the equation $x^2+y^2=1$, we *think* of the functions $y=f(x)$ implicitly defined by the equation. Since y is a function of x, we can differentiate the equation, using the chain rule, obtaining

$$2x+2y\frac{dy}{dx}=0.$$

Solving for $\frac{dy}{dx}$ we then obtain

$$\frac{dy}{dx}=-\frac{x}{y}.$$

We have thus again obtained a formula for $\frac{dy}{dx}$ in terms of the function y itself. And, of course, if we were lucky enough to know a formula for y in terms of x we could substitute it for y at this point. But even in its present form, without eliminating y, we can use the formula for computing the slope of the graph or for other applications. For example the slope of the graph (in this case the circle) at the point $(0,1)$ is $\left.\frac{dy}{dx}\right|_{(0,1)}=-\frac{0}{1}=0$. And the slope at the point $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$ is

$$\left.\frac{dy}{dx}\right|_{(-\frac{1}{2},-\frac{\sqrt{3}}{2})}=-\left(\frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}\right)=-\frac{1}{\sqrt{3}}.$$

Example 4: Use implicit differentiation to find the equation of the tangent line to the graph of $xy^2+x^2y-6=0$ at the point $(1,2)$. (Note, incidently, that this point does indeed lie on the graph .) The equation of the tangent line is $y=2+f'(1)(x-1)$, thus we need only find the slope at the point $(1,2)$. Differentiating the equation implicitly, using the product and chain rules, we obtain

$$1\cdot y^2+x\cdot 2y\frac{dy}{dx}+2x\cdot y+x^2\cdot\frac{dy}{dx}-0=0,$$

and solving for $\frac{dy}{dx}$ we have

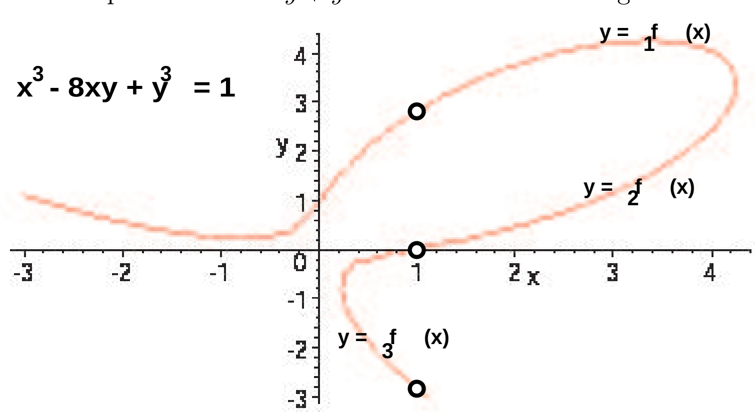
$$\frac{dy}{dx}=\frac{-y^2-2xy}{2xy+x^2}=-\frac{y^2+2xy}{x^2+2xy}.$$

At the point $(1,2)$ the value of the derivative is

$$\left.\frac{dy}{dx}\right|_{(1,2)}=-\frac{4+4}{1+4}=-\frac{8}{5},$$

and hence the tangent line has the equation $y=2-(8/5)(x-1)$.

Example 5: Return to the equation $x^3-8xy+y^3=1$ with which we begin this section.



Although we cannot explicitly solve this equation for y in terms of x, we can find the derivative (of any one of the three functions defined implicitly by the equation). Differentiating implicitly

$$3x^2 - 8y - 8xy' + 3y^2y' = 0,$$

and, solving for y', we have $y' = (8y - 3x^2)/(3y^2 - 8x)$. We can use this formula to find the slope at any point on the curve. Let us try, for example, to find the points on the curve for which $x = 1$. For this we solve the equation $1 - 8y + y^3 = 1$, i.e. $y^3 - 8y = 0$. This equation has three roots, $y = 0$, $\sqrt{8}$, $-\sqrt{8}$. Thus the three points $(1, 0)$, $(1, \sqrt{8})$ and $(1, -\sqrt{8})$ are on the curve. The slope at $(1, 0)$ is

$$y'|_{(1,0)} = \frac{8 \cdot 0 - 3 \cdot 1^2}{3 \cdot 0^2 - 8 \cdot 1} = \frac{3}{8}.$$

Similarly, at $(1, \sqrt{8})$ the slope is

$$y'|_{(1,\sqrt{8})} = \frac{8 \cdot \sqrt{8} - 3 \cdot 1^2}{3 \cdot \left(\sqrt{8}\right)^2 - 8 \cdot 1} = \frac{8\sqrt{8} - 3}{16} \approx 1.22671$$

And, finally, the slope at the point $(1, -\sqrt{8})$ is

$$y'|_{(1,-\sqrt{8})} = \frac{8 \cdot (-\sqrt{8}) - 3 \cdot 1^2}{3 \cdot \left(-\sqrt{8}\right)^2 - 8 \cdot 1} = \frac{-8\sqrt{8} - 3}{16} \approx -1.60171$$

We have, in fact, computed the derivatives $f_1(1)$, $f_2(1)$ and $f_3(1)$. The functions f_1 and f_2 are *increasing* in the vicinity of $x = 1$, and the function f_3 is decreasing. And we learned this in spite of the fact that we have no explicit formulas for the functions. Such is the value of implicit differentiation!

Example 7: Suppose a differentiable function f has an inverse f^{-1}. Find the derivative of f^{-1}. To solve this problem, let $y = f^{-1}(x)$. Then because f and f^{-1} are inverse functions, $f(y) = x$. Hence, differentiating implicitly we have: $f'(y)\frac{dy}{dx} = 1$. Now, solving for $\frac{dy}{dx}$ we get:

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

$$\left[f^{-1}(x)\right]' = \frac{1}{f'(f^{-1}(x))}$$

The above two formulas express the derivative of f^{-1} in terms of the derivative of f.

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**