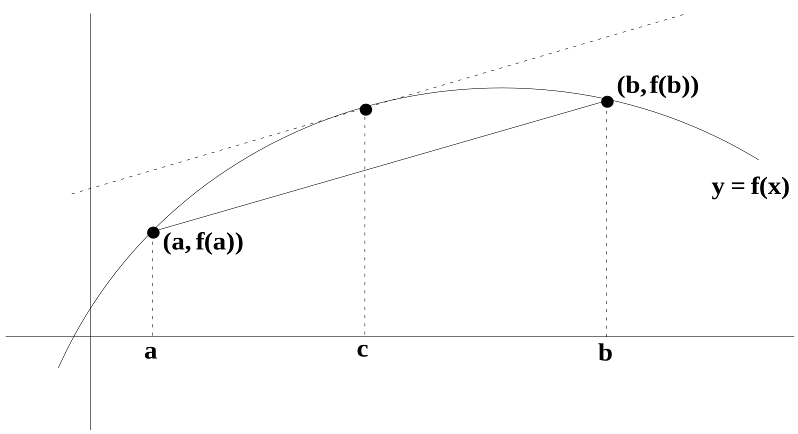
The Mean Value Theorem

The derivative of a function is a powerful tool for analyzing the function's behavior. If $f'(x_0)$ exists at a point x_0, for example, then we not only know that the function is continuous there but also that its graph has a tangent line at $(x_0, f(x_0))$. We have characterized this fact by saying that the graph is "smooth" at the point, i.e. it does not have a "corner". Another way of saying this is to say that the function can be approximated by the linear function $f(x_0) + f'(x_0)(x - x_0)$ in the immediate vicinity of the point. We sometimes call this kind of information about the function "local information" because it tells us how the function behaves at a single point.

We can also ask questions about a function's "global behavior". For example does the function *increase* throughout some interval, or does it *decrease*? Does its graph *rise* as we move from left to right on the x-axis, or does it *fall*? We also expect that the derivative will give us this kind of global information. It does—and the main tool for extending the derivative's influence from individual points to an entire interval is the *Mean Value Theorem*.

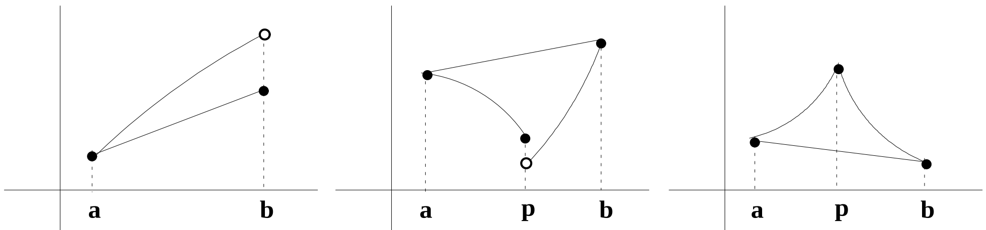
Theorem 1 (The Mean Value Theorem): Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that f' exists on the open interval (a, b). Then there exists a point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



We will provide a proof presently, but first let us be sure to understand just what the theorem says and how it provides the "global" view that we promised. Taken at face value, especially in the presence of the figure, it seems rather obvious. It says, in effect, that there is a place in the interval (a, b) where the tangent line to the graph is parallel to the secant line connecting the end points $(a, f(a))$ and $(b, f(b))$. (Just imagine sliding the secant line upward until it becomes tangent to the graph!) Or, in the language of rates of change, if your *average speed* from the initial to the final toll booths on the New Jersey Turnpike was 75 miles per hour (remember that they stamp the times on the ticket), then there was at least one moment along the way when your instantaneous speed was *exactly* 75 miles per hour. If you try to convince the officer that you were always driving below the speed limit you will lose—the officer, having taken calculus in college, will recite to you the Mean Value Theorem.

Of course the hypotheses of the theorem are essential. If the function is not continuous (it has a jump somewhere) or not differentiable (its graph turns a sharp corner somewhere) then there need be no place where the secant and tangent lines are parallel. (Look at the following figures.)



Discontinuity at an end point — *Discontinuity at an interior point p* — *No derivative at an interior point p*

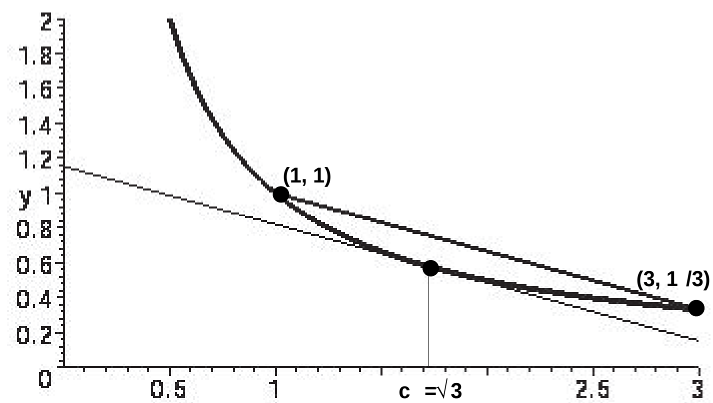
Applet: Mean Value Theorem **Try it!**

Example 1: Consider the function $f(x) = |x|$ on $[-1, 1]$. The Mean Value Theorem does not apply because the derivative is not defined at $x = 0$. Indeed $(|1| - |-1|)/(1-(-1)) = 0$, and there is clearly no value of c for which $f'(c) = 0$.

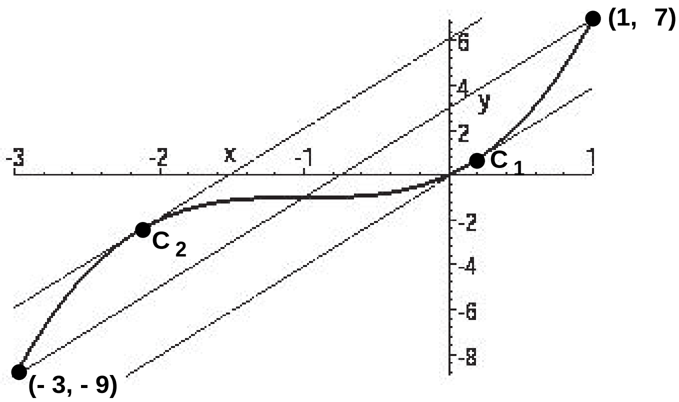
Example 2: Under what circumstances does the Mean Value Theorem apply to the function $f(x) = 1/x$? The only point to avoid is $x = 0$ where the function and its derivative are not defined. Thus we can apply the theorem on any interval $[a, b]$ that does not contain $x = 0$. It does not apply, for example, to the interval $-1 \le x \le 1$. It does apply to to the interval $[1, 3]$. Thus there is a point c in the open interval $(1, 3)$ such that $f'(c) = -1/c^2$ is equal to

$$\frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{3} - \frac{1}{1}}{3 - 1} = -\frac{1}{3}.$$

Solving $-1/c^2 = -1/3$ we have $c = \sqrt{3}$.



Example 3: Verify the conclusion of the Mean Value Theorem for the function $f(x) = (x+1)^3 - 1$ on the interval $[-3, 1]$. First we note that f is continuous on the closed interval $[-3, 1]$ and its derivative $f'(x) = 3(x+1)^2$ is defined in the open interval $(-3, 1)$, thus the Mean Value Theorem applies. The slope of the secant line is $(f(1) - f(-3))/(1-(-3)) = 4$, thus there is at least one point c in the interval where $f'(c) = 4$. Solving $3(c+1)^2 = 4$ we in fact find that there are two such points $c = -1 + (2/3)\sqrt{3}$ and $c = -1 - (2/3)\sqrt{3}$. The two tangent lines that are parallel to the secant line are plotted in the figure.



Let us suppose, now, that a function f is defined on an interval I and that $f'(x) > 0$ for every $x \in I$. This means that the slope of the graph of f is positive everywhere in the interval I. Then the function is *increasing* throughout the interval; i.e. its graph is *rising*. Let us formalize the language we are using here and then prove the statement.

First of all, let us remember that an interval I is the set of real numbers lying between a and b, where a and b are real numbers or $\pm\infty$. The end points a and b may or may not be included in the interval I (of course $\pm\infty$ cannot be included since they are not real numbers). An interval can be *closed*, e.g. $[-5, 5]$. It can be *open*, e.g. $(2, 10)$, $(0, \infty)$, or $(-\infty, \infty)$. Or it can be "half open", e.g. $(3, 5]$, $[0, \infty)$, or $(-\infty, 2]$.

Definition 1: Suppose that f is defined on an interval I, and let x_1 and x_2 denote points in I:

1. f is *increasing on* I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$
2. f is *decreasing on* I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$
3. f is *nondecreasing on* I if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$
4. f is *nonincreasing on* I if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$

Of course "increasing" and "nondecreasing" are very closely related. If a function f is *increasing* its graph is rising (from left to right) whereas if it is *nondecreasing* its graph is generally rising but may have "level plateaus".

Theorem 2: Let I be an interval and let J be the open interval consisting of I minus its endpoints (if any). Suppose that f is continuous on I and differentiable on J. Then

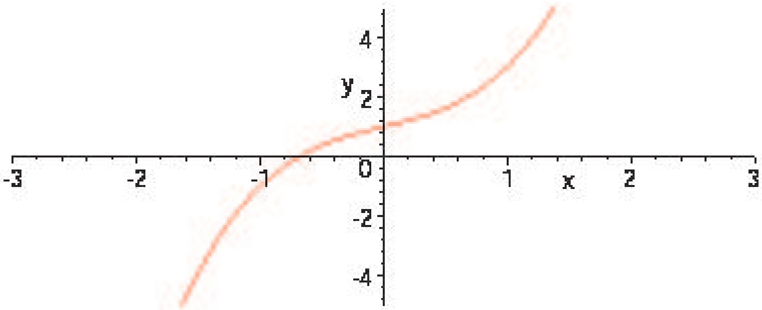
1. If $f'(x) > 0$ for every $x \in J$, then f is *increasing* on I.
2. If $f'(x) < 0$ for every $x \in J$, then f is *decreasing* on I.
3. If $f'(x) \geq 0$ for every $x \in J$, then f is *nondecreasing* on I.
4. If $f'(x) \leq 0$ for every $x \in J$, then f is *nonincreasing* on I.

For a proof we notice that if x_1 and x_2 are any two points in I, $x_1 < x_2$, then the conditions of the Mean Value Theorem are met for the interval $[x_1, x_2]$. Thus there is a point c between x_1 and x_2 such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

Then $f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$ and, since $x_2 - x_1$ is positive, the sign of $f(x_2) - f(x_1)$ is the same as that of $f'(c)$. All four statements of the theorem follow immediately.

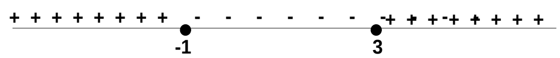
Example 4: On what interval is the function $f(x) = x^3 + x + 1$ increasing (decreasing)? We need to examine the derivative $f'(x) = 3x^2 + 1$. We notice that, in fact, the derivative is positive for all values of x, hence f is increasing on $(-\infty, \infty)$.



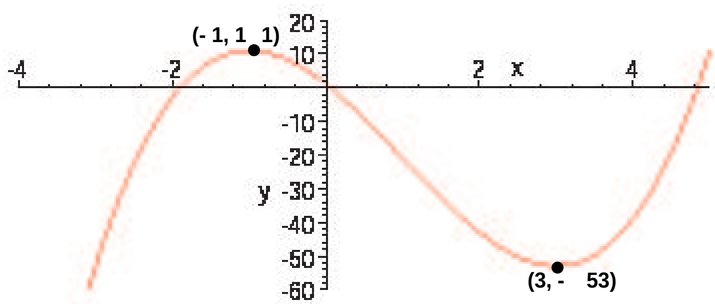
Example 5: Find the intervals on which the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ is increasing and those on which it is decreasing. We examine the derivative

$$f'(x) = 6x^2 - 12x - 18 = 6(x - 3)(x + 1).$$

It is easy to see that the derivative changes sign at the points $x = 3,\ -1$. In the intervals $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$, we record its sign on the following "sign graph":



The function, then, is increasing on $(-\infty, -1)$ and $(3, \infty)$, and it is decreasing on $(-1, 3)$. Computing the values of f at just a few strategic points enables us to sketch a rather accurate graph. The points $(-1, 11)$ and $(3, -53)$ where the derivative changes sign are most useful. The graph "rises" to the first of these points, "falls" between the two points, and "rises" again to the right of the second point. Note these behaviors in the graph below:



Let us return, finally, to a proof of the Mean Value Theorem. Like the Intermediate Value Theorem that we discussed earlier, it depends very much on the deep underlying properties of the real number system, namely the *continuity* of the real line. As such, a correct proof eluded many people in the past. Even Gauss, one of the brightest mathematicians in the eighteenth century, gave a proof of this theorem that subsequently turned out to be deficient. Such historical events make us cautious today, even in the presence of a theorem with such strong intuitive geometric appeal as the Mean Value Theorem.

A correct proof depends, as we noted above, on continuity properties of the real numbers. One such property is that a continuous function takes on a maximum and a minimum value on a closed interval. We state this very fundamental fact as a theorem, and we do not give a proof since it is buried so deeply in the foundations of our number system.

Theorem 3: If f is continuous on a closed interval $[a, b]$, then there is a point c_1 in the interval where f assumes its *maximum* value, i.e. $f(x) \le f(c_1)$ for every x in $[a, b]$, and a point c_2 where f assumes its *minimum* value, i.e. $f(x) \ge f(c_2)$ for every x in $[a, b]$.

Finding the point (or points) where f assumes its maximum and minimum values is an important application of calculus. It sometimes goes under the name "Optimization Theory". The following theorem helps us solve the *maximum-minimum* problem:

Theorem 4: If f is defined in an open interval (a, b) and achieves a maximum (or minimum) value at a point $c \in (a, b)$ where $f'(c)$ exists, then $f'(c) = 0$.

Let us prove this in the case of a maximum value. The proof for a minimum value is similar. If $f(c)$ is a maximum value and $f'(c)$ exists, then $f(x) \leq f(c)$ for all x in (a, b). Then, if $a < x < c$

$$\frac{f(x) - f(c)}{x - c} \geq 0, \qquad \text{so} \quad f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0,$$

and, if $c < x < b$

$$\frac{f(x) - f(c)}{x - c} \leq 0, \qquad \text{so} \quad f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0.$$

It thus follows that $f'(c) = 0$.

It is Theorem 4 that helps us find where a continuous function assumes its maximum and minimum values in a closed interval. The only possible locations for the maximum and minimum values are the so-called *critical points*, interior points where the derivative exists and is zero or where the derivative does not exist, or points that are not interior points of the interval (i.e. end points). In a typical application this narrows the search to a finite number of points that we can calculate.

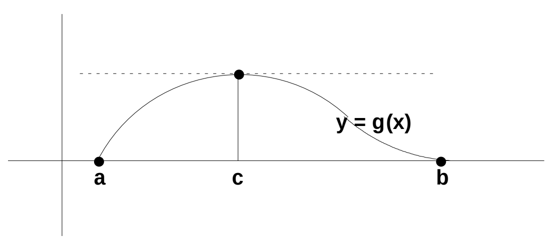
Example 6: Returning to the function $f(x) = 2x^3 - 6x^2 - 18x + 1$ of Example 5, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values. We know that the maximum and minimum values must occur at a critical point or an end point of the interval. The derivative is defined at every point of the interval, thus the only critical points are where $f'(x) = 0$. These are the points $x = -1, 3$. Let us make a small table of the function values at these two points and at the end points of the interval:

x	$f(x)$
-1	11
3	-53
-4	-151
4	-39

These are the only candidates for the maximum or minimum value of $f(x)$ on the interval $[-4, 4]$. Thus the maximum value is 11, occurring at the point $(-1, 11)$. And the minimum value is -151, occurring at the point $(-4, -151)$, one of the end points of the interval. The point $(3, -53)$ is of geometrical interest. The function has a *relative minimum* at this point, although the *absolute* minimum occurs elsewhere, at the left end point of the interval. We will return to these notions again in the context of curve sketching and other applications of the derivative.

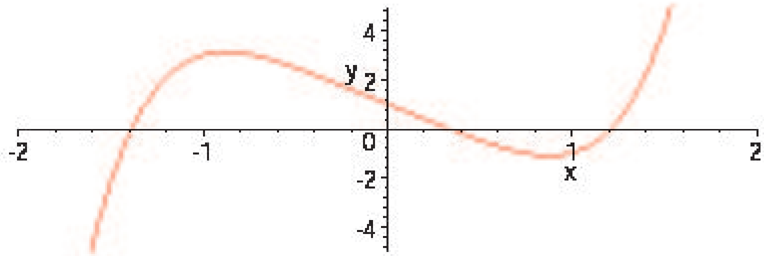
A special case of the Mean Value Theorem is *Rolle's Theorem.* We prove it first and then use it to prove the Mean Value Theorem.

Rolle's Theorem Suppose that the function g is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $g(a) = 0$ and $g(b) = 0$ then there exists a point c in the open interval (a, b) where $g'(c) = 0$.



The proof follows easily from Theorem 4. Assuming that the function g is not identically zero on the interval it must assume either its maximum or minimum value at an interior point c. At such a point $g'(c) = 0$.

Example 7: Use Rolle's Theorem to show that the equation $x^5 - 3x + 1 = 0$ has exactly three real roots. Letting $f(x) = x^5 - 3x + 1$, we have $f'(x) = 5x^4 - 3$. The derivative f' is defined for all real values of x, hence the only critical points of f are where $f'(x) = 0$. There are only two such points, $x = \pm\sqrt[4]{3/5}$. We conclude, therefore, that there can be at most three real roots of the given equation since by Rolle's Theorem there must be a zero of the derivative between every pair of roots. That there are at least three real roots may be deduced from the fact that $f(x)$ is continuous and changes sign at least three times (note the graph below).



Proof of the Mean Value Theorem Let f satisfy the requirements of the mean value theorem. At a point x in the interval $[a, b]$ let $g(x)$ be the vertical distance between the graph of f and the chord connecting the points $[a, f(a)]$ and $[b, f(b)]$ (see the graph accompanying the statement of the Mean Value Theorem). Then the function $g(x)$ is defined by

$$g(x) = f(x) - \left(f(b) + \frac{f(b) - f(a)}{b - a}(x - a) \right),$$

and we notice that the function g satisfies the requirements of Rolle's Theorem, i.e. g is continuous in $[a, b]$, differentiable in (a, b), and $g(a) = g(b) = 0$. Thus there is a point c in the open interval (a, b) where $g'(c) = 0$. Computing $g'(x)$ we have

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Substituting $x = c$ we have $f'(c) - (f(b) - f(a))/(b - a) = 0$, and this is the conclusion of the Mean Value Theorem.

Summary: We can learn much about a function f by studying its derivative. For example we can determine where f is *increasing* and where it is *decreasing*. And we can discover where it assumes its *maximum* and *minimum* values. The key to the relationship between such global properties of f and the behavior of its derivative f' is the Mean Value Theorem. It will often arise in similar circumstances—when we need to connect *local* and *global* behavior of a function. We will turn shortly to the problem of finding "anti derivatives" of functions and to the surprising connection of them to the second major problem of calculus—defining and finding the area of geometrical regions determined by graphs of functions. The Mean Value Theorem will again turn out to play a star role in those new investigations.

Exercises: Problems **Check what you have learned!**

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