

## Derivatives of the Trigonometric Functions

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The trigonometric functions are of fundamental importance in modeling periodic phenomena—light and sound waves, oscillating crystals, time-keeping devices, and a myriad of similar periodic motions. Their derivatives measure the velocity, frequency, and energy of such physical systems and, as a result, occur in differential equations that describe such systems. In this section we develop the essential differentiation rules for sine, cosine, and other trigonometric functions.

We begin with the function  $\sin x$ . Resorting to the limit definition of derivative:

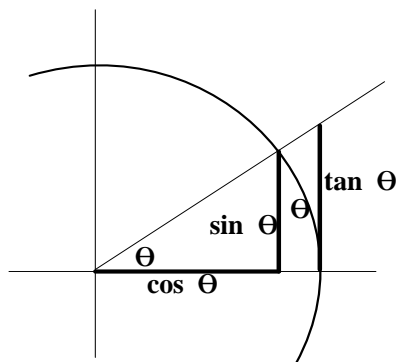
$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} (\sin x) \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} (\cos x) \frac{\sin h}{h} \\ &= (\sin x) \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + (\cos x) \lim_{h \rightarrow 0} \frac{\sin h}{h} \end{aligned}$$

The computation will be complete when we evaluate the two important limits in the final line.

**Theorem 1:**  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ , and  $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$ .

**Proof:** The first of these limits is easily made convincing by calculating the value of  $\sin \theta / \theta$  for some small values of  $\theta$ .

$\theta$	$\sin \theta / \theta$
$10^{-1}$	.99833416646828152307
$10^{-2}$	.99998333341666646825
$10^{-3}$	.99999833333334166667
$10^{-4}$	.999999983333333417
$10^{-5}$	.999999999833333333
$10^{-6}$	.999999999983333333
$10^{-7}$	.999999999998333333
$10^{-8}$	.999999999999833333
$10^{-9}$	.999999999999998333
$10^{-10}$	1.000000000000000000



But no amount of numerical computation constitutes a *proof*. For that we refer to the definition of the trigonometric functions in terms of the unit circle. Refer to the figure, above, in which the angle  $\theta$ , measured in radians, is the length  $s$  of the arc subtended on the unit circle by the central angle  $\theta$ .

First, note that because  $\theta$  radians is to  $2\pi$  radians as the length of the arc is to the circumference, we have (since  $r = 1$ )  $\theta/2\pi = s/2\pi$ , or  $s = \theta$ . Similarly, the area of the sector is to  $\pi r^2 = \pi$  as  $s$  is to  $2\pi$  again;

thus the area of the sector equals  $s/2 = \theta/2$ .

Second, if we let  $y$  be the length of the vertical side of the large triangle, then from similar triangles we have that  $y/\sin \theta = 1/\cos \theta$ . Thus,  $y = \sin \theta / \cos \theta = \tan \theta$ .

The values of  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  are then the lengths of the labeled line segments. From the figure we see that the area of the small triangle is less than or equal to the area of the sector, whose area is less than or equal to the area of the larger outer triangle. Hence, from the formula of the area of the sector developed above, and the lengths of the legs of the two triangles, we have

$$\frac{1}{2} \sin \theta \cos \theta \leq \frac{\theta}{2} \leq \frac{1}{2} \tan \theta$$

$$\frac{1}{2} \sin \theta \cos \theta \leq \frac{\theta}{2} \leq \frac{1}{2} \frac{\sin \theta}{\cos \theta}$$

Multiplying the inequalities by 2 and dividing by  $\sin \theta$ , we have

$$\cos \theta \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}$$

Finally, taking reciprocals we have

$$\frac{1}{\cos \theta} \geq \frac{\sin \theta}{\theta} \geq \cos \theta$$

From this we conclude, finally, that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

since it is “sandwiched” between  $1/\cos \theta$  and  $\cos \theta$ , both of which have the limit 1 as  $\theta \rightarrow 0$ . We should note that the argument just given, based on the figure, assumed that  $\theta > 0$ . Thus, strictly speaking, we have just shown that  $\lim_{\theta \rightarrow 0^+} (\sin \theta)/\theta = 1$ . The left-hand limit must, of course, be the same since  $(\sin \theta)/\theta$  is an even function.

The second of the two limits in the theorem can be obtained from the first. Using known trigonometric identities we have

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} \frac{\cos \theta + 1}{\cos \theta + 1} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)} = \lim_{\theta \rightarrow 0} \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)} \\ &= -\lim_{\theta \rightarrow 0} \sin \theta \cdot \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta + 1} \\ &= -\lim_{\theta \rightarrow 0} \sin \theta \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta + 1} \\ &= -0 \cdot 1 \cdot \frac{1}{2} = 0 \end{aligned}$$

**Example 1:** Evaluate  $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$ . We rewrite the expression so as to recognize the limit of Theorem 1:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 3x}{x} &= \lim_{x \rightarrow 0} 3 \cdot \frac{\sin 3x}{3x} = \\ &= 3 \cdot \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} = 3 \cdot \lim_{u \rightarrow 0} \frac{\sin u}{u} = 3 \cdot 1 = 3 \end{aligned}$$

**Example 2:** The function  $f(x) = \sin x/x$  is defined and continuous at every point except  $x = 0$ . However it has a continuous extension to  $x = 0$  since the  $\lim_{x \rightarrow 0} f(x) = 1$  exists:

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

**Applet: Limit of  $\sin(x)/x$  as  $x$  approaches 0 Try it!**

With the two limits of Theorem 1 evaluated, we now know that the derivative of  $\sin x$  is  $\sin x \cdot 0 + \cos x \cdot 1$ , which we record in the following theorem.

**Theorem 2:**  $\frac{d}{dx} \sin x = \cos x$ , and  $\frac{d}{dx} \cos x = -\sin x$ .

The first is proved. As for  $\frac{d}{dx} \cos x$ , we can use the trigonometric identity of complementary angles to deduce this as follows:

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - x\right) \cdot (-1) = -\sin x.$$

**Example 3:** Differentiate  $\sin 2x$ ,  $\sin(x^2 + 1/x)$ , and  $\cos(3x + \sqrt{x})$ . Using the chain rule in these three examples we have

$$\begin{aligned}\frac{d}{dx} \sin 2x &= \cos 2x \cdot 2 = 2 \cos 2x \\ \frac{d}{dx} \sin\left(x^2 + \frac{1}{x}\right) &= \cos\left(x^2 + \frac{1}{x}\right) \cdot \left(2x - \frac{1}{x^2}\right) = \left(2x - \frac{1}{x^2}\right) \cos\left(x^2 + \frac{1}{x}\right) \\ \frac{d}{dx} \cos(3x + \sqrt{x}) &= -\sin(3x + \sqrt{x}) \cdot \left(3 + \frac{1}{2\sqrt{x}}\right)\end{aligned}$$

**Example 4:** Differentiate  $y = \sin x \cos x$  and  $y = \sin^2(\cos(x^2 + 2))$ . The first function is a product, thus  $y' = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x$ . The second function requires repeated uses of the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= 2 \sin(\cos(x^2 + 2)) \cdot \frac{d}{dx} \sin(\cos(x^2 + 2)) \\ &= 2 \sin(\cos(x^2 + 2)) \cos(\cos(x^2 + 2)) \cdot \frac{d}{dx} \cos(x^2 + 2) \\ &= 2 \sin(\cos(x^2 + 2)) \cos(\cos(x^2 + 2))(-1) \sin(x^2 + 2) \cdot \frac{d}{dx} (x^2 + 2) \\ &= 2 \sin(\cos(x^2 + 2)) \cos(\cos(x^2 + 2))(-1) \sin(x^2 + 2) \cdot (2x) \\ &= -4x \sin(\cos(x^2 + 2)) \cos(\cos(x^2 + 2)) \sin(x^2 + 2)\end{aligned}$$

Differentiation rules for the remaining trigonometric functions are obtained from those of  $\sin x$  and  $\cos x$ :

**Theorem 3:** The derivatives of  $\tan x$ ,  $\cot x$ ,  $\sec x$ , and  $\csc x$  are:

$$\begin{aligned}\frac{d}{dx} \tan x &= \sec^2 x \\ \frac{d}{dx} \cot x &= -\csc^2 x \\ \frac{d}{dx} \sec x &= \sec x \tan x \\ \frac{d}{dx} \csc x &= -\csc x \cot x\end{aligned}$$

The proofs of these rules are by direct calculation, using the product, quotient and power rules:

$$\begin{aligned}\frac{d}{dx} \tan x &= \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \\ \frac{d}{dx} \cot x &= \frac{d}{dx} \frac{\cos x}{\sin x} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x \\ \frac{d}{dx} \sec x &= \frac{d}{dx} \frac{1}{\cos x} = -\frac{(-\sin x)}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x\end{aligned}$$

$$\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

As always the most common applications of these rules is in combination with the chain rule and all the other differentiation rules.

**Example 5:**  $\frac{d}{dx} \sec^2 x = 2 \sec x \cdot \sec x \tan x = 2 \sec^2 x \tan x$ .

**Example 6:** Compute the derivative of  $y = (\sin^3(\tan^2(2x)))^4$ . This example requires six applications of the chain rule:

$$\begin{aligned} y' &= \frac{d}{dx} (\sin^3(\tan^2(2x)))^4 \\ &= 4(\sin^3(\tan^2(2x)))^3 \frac{d}{dx} \sin^3(\tan^2(2x)) \\ &= 4(\sin^3(\tan^2(2x)))^3 \cdot 3 \sin^2(\tan^2(2x)) \frac{d}{dx} \sin(\tan^2(2x)) \\ &= 4(\sin^3(\tan^2(2x)))^3 (3) \sin^2(\tan^2(2x)) \cdot \cos(\tan^2(2x)) \frac{d}{dx} \tan^2(2x) \\ &= 4(\sin^3(\tan^2(2x)))^3 (3) \sin^2(\tan^2(2x)) \cos(\tan^2(2x)) \cdot 2 \tan(2x) \frac{d}{dx} \tan(2x) \\ &= 4(\sin^3(\tan^2(2x)))^3 (3) \sin^2(\tan^2(2x)) \cos(\tan^2(2x)) (2) \tan(2x) \cdot \sec^2(2x) \frac{d}{dx} (2x) \\ &= 4(\sin^3(\tan^2(2x)))^3 (3) \sin^2(\tan^2(2x)) \cos(\tan^2(2x)) (2) \tan(2x) \sec^2(2x) \cdot 2 \\ &= 48(\sin^3(\tan^2(2x)))^3 \sin^2(\tan^2(2x)) \cos(\tan^2(2x)) \tan(2x) \sec^2(2x) \end{aligned}$$

Were you able to keep those six applications of the chain rule straight? It would be understandable if you couldn't. In complicated situations like this it is often helpful to explicitly write the chain of functions that go into the composite function:

$$y = u^4, \quad u = v^3, \quad v = \sin w, \quad w = t^2, \quad t = \tan s, \quad s = 2x$$

Then, according to the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dt} \cdot \frac{dt}{ds} \cdot \frac{ds}{dx} \\ &= 4u^3 \cdot 3v^2 \cdot \cos w \cdot 2t \cdot \sec^2 s \cdot 2 \end{aligned}$$

and, if we now substitute the values of all the intermediate variables back into the final expression, we obtain the same result as before. In simpler examples we had encouraged “thinking the intermediate functions” rather than explicitly writing them down. But in complex cases such as the present one that may leave us befuddled.

**Summary:** In this section we introduced the basic differentiation formulas for each of the trigonometric functions. Only the derivative of  $\sin x$  was computed directly from the limit definition. All the others then followed by using our general differentiation rules, and likewise we can handle through the general rules any more complicated expressions that involve trigonometric functions. It is now time to move on to important applications of differentiation and its important place in modeling real-world problems.

**Exercises: Problems Check what you have learned!**

**Videos: Tutorial Solutions See problems worked out!**