Differentiation Rules

The Power Rule is an example of a *differentiation rule*. For functions of the form x^r, where r is a constant real number, we can simply write down the derivative rather than go through a long computation of the limit of a difference quotient. Developing a repertoire of such basic rules, and gaining skill in using them, is a large part of what calculus is about. Indeed, calculus may be described as the study of elementary functions, a few of their basic properties such as *continuity* and *differentiability*, and a toolbox of computational techniques for computing derivatives (and later integrals). It is the latter computational ingredient that most students recall with such pleasure as they reflect back on learning calculus. And it is skill with those techniques that enables one to apply calculus to a large variety of applications.

Building the Toolbox We begin with an observation. It is possible for a function f to be *continuous* at a point a and not be *differentiable* at a. Our favorite example is the absolute value function $|x|$ which is continuous at $x = 0$ but whose derivative does not exist there. The graph of $|x|$ has a sharp corner at the point $(0, 0)$ (cf. Example 9 in Section 1.3). Continuity says only that the graph has no "gaps" or "jumps", whereas differentiability says something more—the graph not only is not "broken" at the point but it is in fact "smooth". It has no "corner". The following theorem formalizes this important fact:

Theorem 1: If $f'(a)$ exists, then f is continuous at a.

The proof is not difficult. To show that f is continuous at a we must show that $\lim_{x\to a} f(x) = f(a)$, or that $\lim_{h\to 0} f(a + h) = f(a)$. We will accomplish this by showing that $\lim_{h\to 0}(f(a + h) - f(a)) = 0$:

$$\begin{aligned}\lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot h\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\lim_{h\to 0} h = f'(a)\cdot 0 = 0\end{aligned}$$

We have used in the proof only elementary properties of limits concerning sums or products of two functions. Notice that the crucial step was in isolating the difference quotient $(f(a + h) - f(a))/h$, whose limit, $f'(a)$, exists by our assumption.

The theorem confirms our intuition that *differentiability* is a stronger notion than *continuity*. A function can be continuous without being differentiable, but it cannot be differentiable without also being continuous. A function whose derivative exists at every point of an interval is not only continuous, it is *smooth*, i.e. it has no sharp corners.

We now proceed to develop differentiation rules. Cognizant of the way functions are built from a small number of simple functions using algebraic operations and composition, we examine how differentiation regards these operations. In the theorems that follow we assume that f and g are functions whose derivatives f' and g' exist.

Theorem 2: Suppose $y = f(x)$ is a function that has derivative f'. Then, $(cf)' = cf'$, where c is a constant. Or in Leibniz's notation $\frac{d}{dx}(cf(x)) = c \cdot \frac{d}{dx}f(x)$.

A proof simply uses the corresponding property of limits:

$$\frac{d}{dx}(cf(x)) = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\cdot f'(x).$$

Just as constants can be moved outside a limit, so they can be moved outside the operation of differentiation.

Example 1: $(3x^2)' = 3(x^2)' = 3 \cdot 2x = 6x$. In Leibniz's notation $\frac{d}{dx}(3x^2) = 3 \cdot \frac{d}{dx}(x^2) = 3 \cdot 2x = 6x$.

Theorem 3: If f and g are functions with derivatives f' and g', respectively, then $(f + g)' = f' + g'$. In words, the derivative of a sum is the sum of the derivatives.

Again, this follows immediately from the corresponding property of limits:

$$\begin{aligned}\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)\end{aligned}$$

In fact it follows that the derivative of any number of terms is the sum of the derivatives of each term. For example $(f+g+h)' = ((f+g)+h)' = (f+g)'+h' = f'+g'+h'$.

Example 2: Theorems 2 and 3 taken together enable us to differentiate any polynomial. For example

$$\begin{aligned} \frac{d}{dx}(3x^2+2x+7) &= \frac{d}{dx}(3x^2)+\frac{d}{dx}(2x)+\frac{d}{dx}(7) = \\ &= 3\cdot\frac{d}{dx}(x^2)+2\cdot\frac{d}{dx}(x)+\frac{d}{dx}(7) \\ &= 3\cdot 2x+2\cdot 1+0 = 6x+2. \end{aligned}$$

And, similarly

$$\frac{d}{dx}(x+\sqrt{x}) = 1+\frac{d}{dx}x^{1/2} = 1+(1/2)x^{-1/2} = 1+\frac{1}{2\sqrt{x}}.$$

Theorem 4 (The Product Rule): If f and g are functions with derivatives f' and g', respectively, then $(fg)' = fg'+gf'$. In words, "the derivative of a product is the first factor times the derivative of the second, plus the second factor times the derivative of the first".

It is, in fact, useful to learn to state the theorem in words. Comparing a given example to the mathematical statement is prone to error, whereas carrying out the necessary computations while reciting the rule is a convenient skill to learn. We look at several examples of the rule in action and then provide a proof.

Example 3: Find $f'(x)$ in two ways, given $f(x) = (5x+3)(x+2)$. The first way, of course, might be to multiply out the given expression and then differentiate the resulting polynomial: $[(5x+3)(x+2)]' = (5x^2+13x+6)' = 10x+13$. Using the product rule we get

$$\begin{aligned} [(5x+3)(x+2)]' &= (5x+3)(x+2)'+(x+2)(5x+3)' \\ &= (5x+3)\cdot 1+(x+2)\cdot 5 \\ &= 5x+3+5x+10 \\ &= 10x+13. \end{aligned}$$

Example 4: If $y = \sqrt{x}(x^2+2)$, find $\frac{dy}{dx}$. Using the product rule, this time carrying out the computations as we recite the rule:

$$\frac{dy}{dx} = \sqrt{x}\cdot 2x+(x^2+2)\frac{1}{2\sqrt{x}}.$$

Students always ask the question "Must I simplify this?" The answer is "yes" if you know what you want to do with the derivative or what other results it needs to be compared with. This is normally the case. However, note that there is no unique "simplest" form. It definitely does depend on what use you intend to make of the result. A reasonable simplification in this case might be $y' = 2x^{3/2}+(1/2)x^{3/2}+(1/2)x^{-1/2} = (5/2)x^{3/2}+(1/2)x^{-1/2}$.

To give a proof of the product rule, we start start with the limit we must evaluate and then subtract and add $f(x+h)g(x)$ to pull the products apart:

$$\begin{aligned} (fg)'(x) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)(g(x+h)-g(x))+g(x)(f(x+h)-f(x))}{h} \\ &= \lim_{h\to 0}\left(f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right) \\ &= f(x)g'(x)+g(x)f'(x) \end{aligned}$$

Note that we have used the fact that a differentiable function is continuous to conclude that $\lim_{h\to 0} f(x+h) = f(x)$.

Theorem 5 (The Reciprocal Rule): Suppose f has derivative f'. Then for any x such that $f(x) \neq 0$, $\left(\frac{1}{f}\right)'(x) = -\frac{f(x)'}{f(x)^2}$. That is, $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$.

Again, this follows from the limit definition:

$$\begin{aligned}
\left(\frac{1}{f}\right)'(x) &= \lim_{h\to 0} \frac{\left(\frac{1}{f}\right)(x+h) - \left(\frac{1}{f}\right)(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x)f(x+h)} \\
&= -\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \frac{1}{f(x+h)} \frac{1}{f(x)} \\
&= -\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} \frac{1}{f(x+h)} \lim_{h\to 0} \frac{1}{f(x)} \\
&= -f'(x)\frac{1}{f(x)}\frac{1}{f(x)} = -\frac{f'(x)}{f(x)^2}
\end{aligned}$$

In evaluating the three limits, we recognized the first as the definition of $f'(x)$. In the second we used the continuity of f at x (Theorem 1). And the third was independent of h.

Example 5: Let $f(x) = \frac{1}{x^2+1}$. Then $f'(x) = -\frac{2x}{(x^2+1)^2}$. Again we wrote this down while we recited the rule "minus the derivative of the denominator divided by the square of the denominator".

Theorem 6 (The Quotient Rule): Suppose f and g have derivatives f' and g', respectively. Then for any x such that $g(x) \neq 0$, $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f(x)' - f(x)g(x)'}{g(x)^2}$. That is, $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$. In words, "the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the denominator squared".

The quotient rule is really just the product and reciprocal rules combined, for

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f \cdot \left(-\frac{g'}{g^2}\right) + \frac{1}{g} \cdot f' = \frac{gf' - fg'}{g^2}.$$

Example 6: $f(x) = \frac{x+1}{x+2}$. Then, writing as we recite the rule:

$$f'(x) = \frac{(x+2)(1) - (x+1)(1)}{(x+2)^2} = \frac{x+2-x-1}{(x+2)^2} = \frac{1}{(x+2)^2}.$$

Example 7: $f(x) = \frac{1+\sqrt{x}}{x^2+3x+2}$. Then

$$f'(x) = \frac{(x^2+3x+2)\frac{1}{2\sqrt{x}} - (1+\sqrt{x})(2x+3)}{(x^2+3x+2)^2}.$$

It would try one's patience to obtain this result using the limit definition instead of the quotient rule. Should we simplify it? No, unless we know what use is to be made of it.

Example 8: For $f(x) = \frac{1}{x} = x^{-1}$, find the derivative three ways, using the power rule, the reciprocal rule, and the quotient rule:

Power Rule: $f'(x) = (-1)x^{-2} = -\frac{1}{x^2}$

Reciprocal Rule: $f'(x) = -\frac{1}{x^2}$

Quotient Rule: $f'(x) = \frac{x\cdot 0 - 1\cdot 1}{x^2} = -\frac{1}{x^2}$

The collection of rules that we now have enable us to write down the derivatives of a remarkable variety of functions, knowning only the derivatives of a few basic functions. There is one situation not covered by our rules, however, namely how do we deal with the composition of functions? How would we differentiate $\sqrt{x^2+1}$, for example? We will add one final rule to our arsenal to handle functions that are built up by the operation of composition, the so-called *chain rule*. It is perhaps the most important differentiation rule of all.

Theorem 7 (The Chain Rule): Let $(f \circ g)(x) = f(g(x))$ be the function defined from f and g by composition. Assume that g is differentiable at the point x and that f is differentiable at the point $g(x)$. Then the composite function $f \circ g$ is differentiable at the point x, and

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x).$$

Leibniz' notation gives us a useful alternative way to write the chain rule. If we define $u = g(x)$, we can write the composition as the "chain" of functions $y = f(u)$, where $u = g(x)$. Then the chain rule takes the form

$$\frac{dy}{dx} = f'(u)\frac{du}{dx} = \left.\frac{dy}{du}\right|_{u=g(x)} \cdot \frac{du}{dx}, \qquad \text{or simply } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

if we remember that the first factor $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Example 9: For the function $\sqrt{x^2+1}$ we would have, applying the first statement of the chain rule,

$$\left[\sqrt{x^2+1}\right]' = \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

To apply the second form of the rule we write $y = \sqrt{u}$, where $u = x^2 + 1$; then we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

Leibniz' notation really comes into full bloom in writing the chain rule in the second form, above. Remembering that derivatives measure *rate of change*, we interpret $\frac{dy}{du}$ as measuring how much faster y changes than u, and $\frac{du}{dx}$ as measuring how much faster u changes than x. Thus it seems perfectly natural that $\frac{dy}{dx}$ should be the product of these two derivatives, measuring how much faster y changes than x. (If y changes twice as fast as u, and u changes three times as fast as x, then y is changing six times as fast as x.)

Before coming back to a proof of the chain rule we consider a few more examples that illustrate the ease of its use in practice.

Example 10: Differentiate $y = (x^2+2)^{10}$. Writing this as $y = u^{10}$, where $u = x^2 + 2$, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 10u^9 \cdot 2x = 20x(x^2+2)^9.$$

As with nearly any rule, after gaining some facility with its use one can find shortcuts. Instead of explicitly substituting $u = g(x)$ one simply "thinks it" instead. To different $f(g(x))$, then, one "thinks" of the inside function $g(x)$ as an indivisible *glob*, and recites "take the derivative of f with respect to *glob* and then multiply by the derivative of *glob* with respect to x". In this way the derivatives of many composite functions may be written down directly as one recites the rule.

Example 11: Differentiate $f(x) = (1 + 3\sqrt{x})^{35}$. Thinking of $1 + 3\sqrt{x}$ as the *glob* in this case, we think, and write, immediately

$$f'(x) = 35(glob)^{34} \cdot \frac{d}{dx}(glob) = 35(1+3\sqrt{x})^{34} \cdot 3\frac{1}{2\sqrt{x}}$$

(Of course, we normally don't reveal the "*glob*" part outside the family.)

Example 12: For $f(x) = \left(\frac{x+1}{x^2+1}\right)^3$, we have

$$f'(x) = 3\left(\frac{x+1}{x^2+1}\right)^2 \cdot \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2}.$$

Simplify? Sure, go ahead.

Let us prove the Chain Rule: Assume that $y = f(g(x))$, that g is differentiable at x_0, and f is differentiable at $g(x_0)$. Then we must show that $f(g(x))$ is differentiable at x_0 and that $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$. As usual we begin with the limit definition of the derivative at x_0:

$$(f \circ g)'(x_0) = \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0},$$

where we must show that the limit exists and has the given value. Can we rewrite the difference quotient in a more transparent form? A naive (and not completely correct) first step might be to multiply and divide by $g(x) - g(x_0)$ as follows:

$$\frac{f(g(x)) - f(g(x_0))}{x - x_0} = \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \cdot \frac{g(x) - g(x_0)}{x - x_0}$$

The second factor we immediately recognize to be the difference quotient for g, whose limit as $x \to x_0$ is $g'(x_0)$. And, feeling on a roll, we notice that the first factor is also a difference quotient of sorts—it is the slope of a secant line to the graph of f at the point $g(x_0)$. As such, when $x \to x_0$, $g(x) \to g(x_0)$ (g is continuous at x_0), and the quotient should approach the slope $f'(g(x_0))$ of the graph of f. This would yield exactly the expression $f'(g(x_0))g'(x_0)$ that we want.

What, if anything, is wrong with the above "proof"? The one sticky point is that we multiplied *and divided* by $g(x) - g(x_0)$, and this would be a problem if we were dividing by zero. Could $g(x) - g(x_0) = 0$ for values of x different from x_0? And could this happen for values of x arbitrarily close to x_0? The bad news is that it *can*, even though such circumstances are rare. But it takes only *one* exception to render our proof invalid. The good news is that we can fix the problem by taking a slightly closer look at the argument we gave above.

We begin with our assumption that f is differentiable at the point $u_0 = g(x_0)$, i.e. that $\lim_{u \to u_0}(f(u) - f(u_0))/(u - u_0)$ exists and is equal to $f'(u_0)$. Let us introduce the function

$$Q(u) = \begin{cases} \frac{f(u) - f(u_0)}{u - u_0} & \text{if } u \neq 0 \\ f'(u_0) & \text{if } u = u_0 \end{cases}$$

and notice that it is simply the continuous extension of the difference quotient to the point u_0. I.e. $\lim_{u \to u_0} Q(u) = f'(u_0) = Q(u_0)$. Notice also that $f(u)(u - u_0) = f(u) - f(u_0)$. (If $u \neq u_0$ this is immediate from the definition of Q, and if $u = u_0$ it is obvious since both sides of the equation are zero.) In particular, $f(g(x)) - f(g(x_0)) = f(u) - f(u_0) = Q(u)(u - u_0) = Q(g(x))(g(x) - g(x_0))$, and we can return to our initial argument:

$$\begin{aligned} (f \circ g)'(x_0) &= \lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} \frac{Q(g(x))(g(x) - g(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} Q(g(x)) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= Q(g(x_0))g'(x_0) \\ &= Q(u_0)g'(x_0) = f'(u_0)g'(x_0) = f'(g(x_0))g'(x_0) \end{aligned}$$

In concluding that $\lim_{x \to x_0} Q(g(x)) = Q(g(x_0))$ we made key use of the continuity of g at x_0 and Q at $u_0 = g(x_0)$.

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