

The Derivative

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The tangent line problem has been solved. Given a function f and a point x_0 in its domain, the tangent line to the graph of f at the point $(x_0, f(x_0))$ is given by $y = f(x_0) + m(x - x_0)$, where $m = \lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0))/h$, provided the limit exists. The quotients $(f(x_0 + h) - f(x_0))/h$ represent the average rate of change of f over the interval $[x_0, x_0 + h]$.

The method is quite general, the point x_0 being *any* point in the domain of f . Thus a simple change in our point of view allows us to focus on the *rule* whereby the slope m is computed from any point x in the domain of f . The new function defined by this rule is called the *derivative function*, or simply the *derivative of f* :

Definition 1: The *derivative* of a function f is a new function defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of f' is the set of points x where this limit exists, i.e. the set of points where the graph of f has a tangent line. The equation of the tangent line at the point $(x_0, f(x_0))$ is $y = f(x_0) + f'(x_0)(x - x_0)$.

We will say that a function f is *differentiable* at a point $x = a$ if the derivative function f' exists at a .

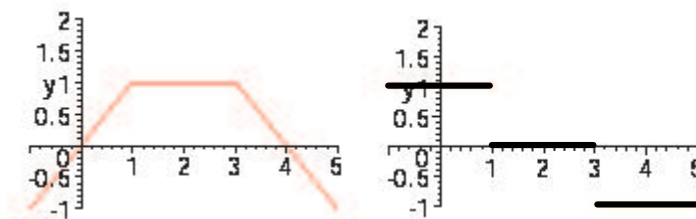
Example 1: Suppose we consider the piecewise defined function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$

Let us find the derivative function f' . This is an especially simple case, the graph of f consisting of three pieces of straight lines. Since $f'(x)$ represents the slope of the graph at x (i.e. the slope of the tangent line to the graph), we know immediately that

$$f'(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

At $x = 1$ and $x = 3$ the derivative is not defined; therefore, f is not differentiable at those points. The graph has sharp “corners” at those two points. At $x = 1$ the slope “jumps” from 1 on the left to 0 on the right. And at $x = 3$ it jumps from 0 on the left to -1 on the right. The derivative function f' is not continuous at those points.



Examples of some commonly encountered functions follow:

Example 2: $f(x) = k$, where k is a constant. Then the graph of f is a horizontal straight line, with slope zero at every point. Thus $f'(x) = 0$ for all x . We notice that this is exactly the result we obtain using Definition 1:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{k - k}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

Example 3: (Linear function) $f(x) = ax + b$, a, b constants. Then $f'(x) = a$, since the graph is a straight line with slope a . Again, using Definition 1 we obtain the same result:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[a(x+h) + b] - [ax + b]}{h} = \lim_{h \rightarrow 0} \frac{ah}{h} = a$$

Example 4: (The derivative of x^2) For $f(x) = x^2$, we have

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

Example 5: (The derivative of x^3) For $f(x) = x^3$, we have

$$\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2.$$

Example 6: (The derivative of $1/x$) For $f(x) = 1/x$, we have

$$\lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.$$

Example 7: (The derivative of \sqrt{x}) For $f(x) = \sqrt{x}$, we have

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Several of the previous examples are special cases of the following general rule:

Theorem 1 (The Power Rule): Suppose that $f(x) = x^r$, where r is any real number. Then $f'(x) = rx^{r-1}$.

It is immediately clear that Examples 4 and 5 verify this rule. In Example 6, if we write $1/x = x^{-1}$, we see that the rule also applies and yields the result $(-1)x^{-2} = -1/x^2$. And in Example 7, writing $\sqrt{x} = x^{1/2}$, the rule yields $(1/2)x^{-1/2}$, which is the same result we obtained using the “rationalization” trick.

We will verify a number of special cases of the Power Rule as we develop appropriate tools, and eventually, when we have precisely defined the general power function x^r , we will prove the rule in its most general form.

Example 8: Find an equation of the tangent line to the graph of $f(x) = x^{4/3}$ at the point where $x = 1$. We know that the desired equation can be written as $y = f(1) + f'(1)(x - 1) = 1 + f'(1)(x - 1)$, thus we need only compute $f'(1)$. From the Power Rule we have $f'(x) = (4/3)x^{4/3-1} = (4/3)x^{1/3}$. So, $f'(1) = 4/3$, and the equation of the tangent line is $y = 1 + (4/3)(x - 1)$.

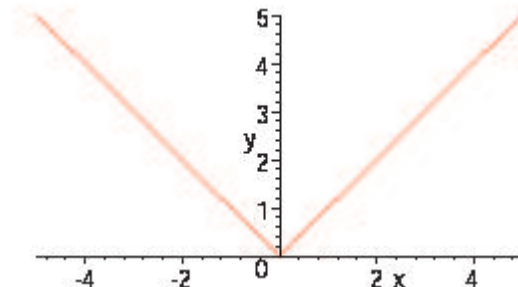
Example 9: Find the derivative of $f(x) = |x|$. We notice immediately that $f'(x) = -1$ when $x < 0$, and that $f'(x) = 1$ when $x > 0$. What happens at $x = 0$? It should be clear from the graph of f , which has a sharp corner at $x = 0$, that $f'(0)$ is not defined. Indeed

$$\lim_{h \rightarrow 0^+} \frac{|0+h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1$$

and

$$\lim_{h \rightarrow 0^-} \frac{|0+h| - 0}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1.$$

Since the right-hand and left-hand limits of the difference quotient are not the same the limit does not exist. I.e. $f'(0)$ is not defined, and hence there is no tangent line to the graph at the point $(0, 0)$.



Example 10: Consider the function f defined by

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

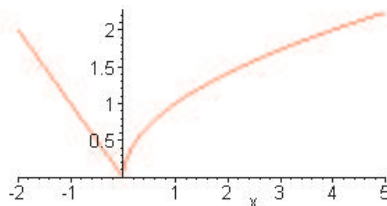
and determine if f has a derivative at all points of its domain. We can see that two applications of the Power Rule yield that if $x < 0$, then $f'(x) = -1$, and if $x > 0$, then $f'(x) = \frac{1}{2\sqrt{x}}$. So, it remains to decide what happens when $x = 0$. To determine this, we need to calculate the limit of the slopes of the secant lines from both the left and right of $x = 0$. Indeed

$$\lim_{h \rightarrow 0^-} \frac{-(0+h) - 0}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$$

and

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt{h}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty.$$

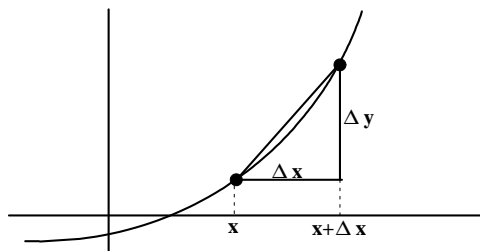
Since the right-hand and left-hand limits of the difference quotients are not the same, the limit does not exist at $x = 0$. Moreover, the fact that the limit of the slopes of the secant lines from the right is infinity indicates that there is a vertical tangent line as we approach $x = 0$ from the right. And of course, there is a tangent line of slope -1 as we approach from the left. So, $f'(0)$ is not defined, and hence there is no unique tangent line to the graph at the point $(0, 0)$.



Notation for the Derivative: Thus far we have denoted the derivative of f as f' . Corresponding to our many ways of writing functions, for example the function $f(x) = x^2$ may be written as $y = x^2$, x^2 , or simply y , we can also write the derivative of f in a number of ways:

$$y' = D_x y = D_x x^2 = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x)$$

All of these expressions can be read as “take the derivative of the function f ”. The notation $\frac{dy}{dx}$ is especially interesting. It was introduced by Leibniz to suggest that the derivative is a limit of a difference quotient.



Writing Δx instead of h to denote a “small” change in x , and $\Delta y = f(x + \Delta x) - f(x)$ to denote the corresponding change in y , then the difference quotient (slope of the secant line) is $\frac{\Delta y}{\Delta x}$. As Δx becomes small the quotient $\frac{\Delta y}{\Delta x}$ becomes a better and better approximation for the slope of the graph at $(x, f(x))$, and in the limit as $\Delta x \rightarrow 0$ approaches the value of $f'(x)$.

The notation $f'(x)$ lends itself easily to denoting the value of the derivative at a particular point $x = x_0$, namely as $f'(x_0)$. When using the notation $\frac{dy}{dx}$ for the derivative, we will customarily denote its value at x_0 by $\frac{dy}{dx} \Big|_{x=x_0}$.

Example 11: For the function $y = f(x) = 1/x$, find the slope of its tangent line at $x = 2$. Compare it with the average rate of change over the interval $[2, 3]$. Using the notations introduced above we can write

$$f'(2) = \left. \frac{dy}{dx} \right|_{x=2} = \left. \frac{d}{dx} \left(\frac{1}{x} \right) \right|_{x=2} = \left. \frac{d}{dx} (x^{-1}) \right|_{x=2} = \left. \left(-\frac{1}{x^2} \right) \right|_{x=2} = -\frac{1}{2^2} = -\frac{1}{4}$$

The notation of Leibniz is very convenient and flexible, and we will make heavy use of it. The average rate of change, or the slope of the secant line, on the interval $[2, 3]$ is $\frac{\Delta y}{\Delta x}$ or $(1/3 - 1/2)/(3 - 2) = -1/6$.

Higher Order Derivatives: We can extend our notation to the case of repeated differentiation. When we differentiate a function $f(x)$ we obtain a new function $f'(x)$. The derivative is again a candidate for differentiation, and we call its derivative *the second derivative of $f(x)$* . So long as the derivatives exist we can continue this process to obtain a succession of higher derivatives. There are a variety of notations for higher derivatives just as there are many alternative notations for y' . Depending on the circumstances we may denote the second derivative, for example, as one of the following:

$$y'' = f''(x) = \frac{d^2 y}{dx^2} = \frac{d}{dx} \frac{d}{dx} f(x) = \frac{d^2}{dx^2} f(x) = D_x^2 y = D_x^2 f(x).$$

Third derivatives may be denoted by

$$y''' = f'''(x) = \frac{d^3 y}{dx^3} = \frac{d^3}{dx^3} f(x) = D_x^3 y = D_x^3 f(x),$$

and in general, the n^{th} derivative, where n is a positive integer, may be denoted by

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x) = D_x^n y = D_x^n f(x),$$

Just as the first derivative has many interpretations that lead to applications—rate of change, slope of a curve, velocity—so do higher derivatives. The second derivative, for example, can represent acceleration—the rate of change of velocity. And we will see, in the next chapter, that it has an important geometrical interpretation related to graphs—concavity, or curvature. Third and fourth derivatives occur in describing engineering problems related to the elastic bending of beams.

Summary: The tangent line problem led us to define the slope of a curve at a point in terms of the limit of slopes of secant lines. This culminated in the notion of *derivative of a function* $y = f(x)$ — a new function $y' = f'(x)$ that gives the slope at any point x . We have illustrated the computation of derivatives graphically (in simple cases), through the direct application of the limit definition (Definition 1), and through the use of rules that can greatly speed the computation of derivatives (e.g. the Power Rule). In the next sections we will greatly expand our repertoire of differentiation rules to provide powerful ways of quickly computing derivatives. And, armed with such power, we will tackle a variety of applications of the derivative to problems in science, economics, and other fields.

Exercises: [Problems](#) **Check what you have learned!**

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