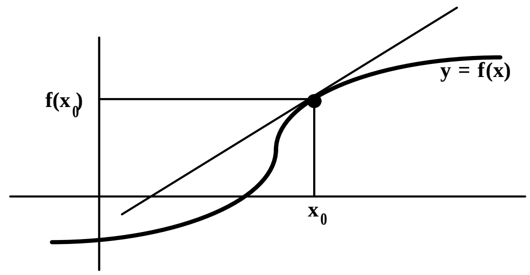
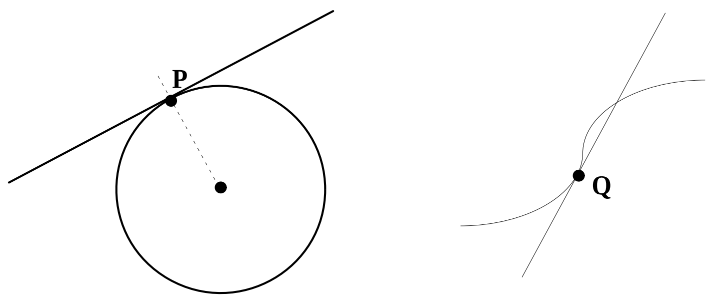
Tangent Lines and Their Slopes

The Tangent Line Problem Given a function $y = f(x)$ defined in an open interval and a point x_0 in the interval, define the tangent line at the point $(x_0, f(x_0))$ on the graph of f.

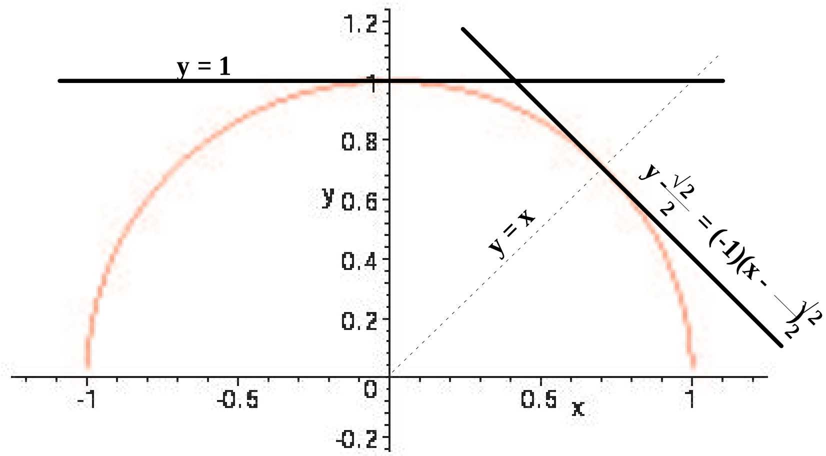


With this problem we begin our study of calculus. Indeed one can say that there are two fundamental problems in calculus—the tangent line problem, and the problem of calculating areas. The first problem leads to the study of the *derivative* and *differential calculus*. The second problem leads to *integral calculus*.

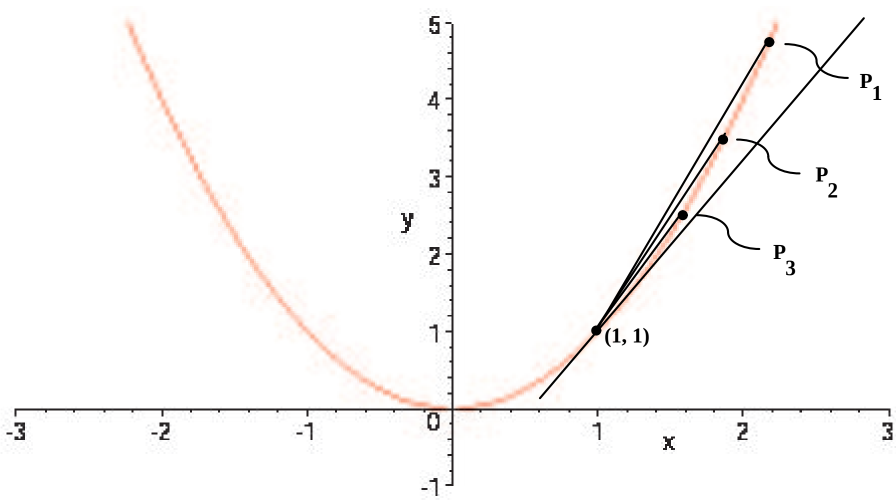
Now the problem of finding the tangent line to a curve has already arisen in geometry. There the tangent to a circle is defined as a line that intersects the circle in exactly one point P. It "touches" the circle at the point P and is perpendicular to a diameter of the circle through P. This definition suffices for the very special case of a circle, but for a more general curve it is not a satisfactory definition. We neither know how to construct a line perpendicular to the curve nor is it the case that a tangent line intersects the curve in only one point (see the figure below where the tangent line at Q intersects the curve more than once). Somehow the manner in which a tangent line intersects a curve is a very local phenomenon at the point of tangency Q—what the line does further from the point Q does not matter. It is no surprise, therefore, that a satisfactory definition of tangent line will involve the notion of limit.



Example 1: Find the equations of the tangent lines to the graph of $f(x) = \sqrt{1-x^2}$ at the points $(0,1)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Let us note, first of all, that the graph of f is a semi-circle and that the given points are, indeed, on the circle. Thus we can use our special knowledge of circles to solve the problem. The tangent line at $(0,1)$ is perpendicular to the y-axis, hence has the equation $y = 1$. And the tangent line at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ is perpendicular to the line $y = x$, hence has slope $m = -1$. Using the point-slope form $y - y_0 = m(x - x_0)$ for the equation of a line through (x_0, y_0) with slope m, we can write the equation of the second tangent line as $y - \frac{\sqrt{2}}{2} = (-1)(x - \frac{\sqrt{2}}{2})$.



Example 2: Let $f(x) = x^2$. Find the equation of the tangent line to the graph of f at the point $(1,1)$. This time we know nothing special about the



geometry of the curve, so we adopt a different procedure. Let us choose several points P_1, P_2, and P_3 on the curve and draw the secant lines from these points to the given point $(1,1)$. (See the figure.) It seems reasonable to assume that these secant lines approximate the tangent line at $(1,1)$, the approximation being better as the point P_i approaches $(1,1)$. Thus we might assume that the slopes of the secant lines approach the slope of the tangent line in the limit as P_i approaches $(1,1)$. This would give us a procedure for calculating the slope of the tangent line: namely let $P = (1+h, f(1+h))$ be a point on the graph near $(1,1)$, calculate the slope of the secant line from P to $(1,1)$, and take the limit as $h \to 0$. Carrying this out we obtain

$$\begin{aligned}
\text{Slope of tangent line} &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{(1+h)^2-1}{h} \\
&= \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2
\end{aligned}$$

This procedure is the basis of our formal definition:

Definition 1: Given a function f and a point x_0 in its domain, the slope of the tangent line at the point $(x_0, f(x_0))$ on the graph of f is

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

if this limit exists. The quotient in this limit (the slope $(f(x_0+h)-f(x_0))/h$ of the secant line) is called the *difference quotient*.

Note that if we let $x = x_0 + h$, then $h = x - x_0$ and we can write the above limit in the equivalent form

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

Example 3: Given $f(x) = \sqrt{x}$, find the equation of the tangent line at $x = 4$. We are seeking the line through the point $(4, 2)$ with slope given by the limit of the difference quotient (Definition 1). Thus the slope is

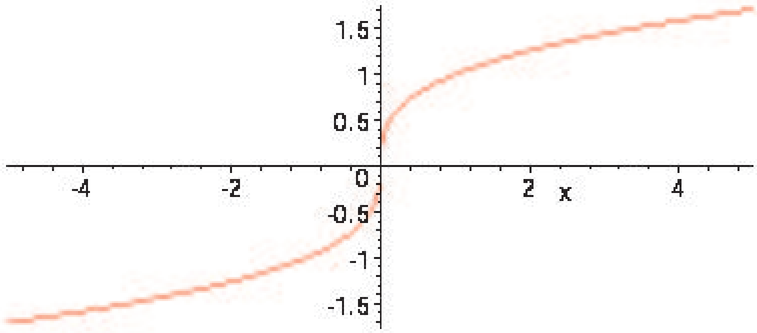
$$\begin{aligned}
\text{Slope of tangent line} &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\
&= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = 1/4
\end{aligned}$$

The slope of the tangent line is $1/4$, so its equation is $y - 2 = \frac{1}{4}(x-4)$. (Note the common trick employed to "simplify" the difference quotient—multiplying numerator and denominator by an expression chosen to rationalize the numerator.)

Example 4: Apply the method of Definition 1 to find the tangent line to the graph of $f(x) = x^{1/3}$ at $x = 0$. In this case the tangent line would be $y - 0 = m(x - 0)$, or $y = mx$, where

$$m = \lim_{h\to 0} \frac{(0+h)^{1/3} - 0^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty$$

However the limit does not exist, thus the slope of the tangent line does not exist. In this case the tangent line does exist but it is vertical. Its equation is thus $x = 0$ (which is not of the form $y = mx$).



Example 5: Let f be the piecewise defined function

$$f(x) = \begin{cases} 2 - x^2 & x \leq 1 \\ x^3 & x > 1 \end{cases}$$

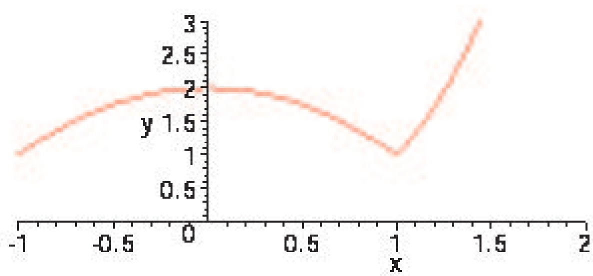
Is the function continuous, and does it have a tangent line at $x = 1$? We first notice that the function is continuous at $x = 1$ because the right-hand limit, left-hand limit, and $f(1)$, all equal 1. We must also treat the difference quotient in two cases depending on whether h is positive or negative. The limit of the difference quotient from the left is

$$\lim_{h \to 0^-} \frac{[2 - (1+h)^2] - 1}{h} = \lim_{h \to 0^-} \frac{-2h - h^2}{h} = \lim_{h \to 0^-} (-2 - h) = -2$$

and from the right the limit is

$$\lim_{h \to 0^+} \frac{(1+h)^3 - 1}{h} = \lim_{h \to 0^+} \frac{3h + 3h^2 + h^3}{h} = \lim_{h \to 0^+} (3 + 3h + h^2) = 3$$

The right and left limits are not equal, hence the slope is undefined. In this case there is no unique tangent line—the graph has a sharp corner at $(1, 1)$.



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