Limits at Infinity and Infinite Limits

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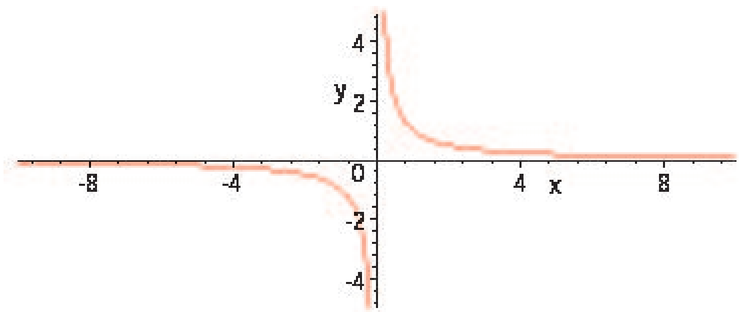
It may be argued that the notion of *limit* is the most fundamental in calculus— indeed, calculus begins with the study of functions and limits. In many respects it is an intuitive concept. The idea of one object "approaching" another, even approaching it "arbitrarily closely" seems natural enough. Our language is full of words that express exactly such actions. And in mathematics the idea of the value of x approaching some real number L, and coming as close to it as we please, does not seem to be stretching the ideas of everyday speech. It is surprising, then, that there are such subtle consequences of the concept of limit and that it took literally thousands of years to "get it right".

In the definition of $\lim_{x\to a} f(x)$, a is a real number. It would be natural to relax this to include the cases $x \to \infty$ and $x \to -\infty$. This means that the value of x increases beyond all bounds that you might name ($x \to \infty$) or decreases below all (negative) bounds that you might name.

Definition 1: $\lim_{x\to\infty} f(x) = L$ means that the value of $f(x)$ approaches L as the value of x approaches $+\infty$. This means that $f(x)$ can be made as close to L as we please by taking the value of x sufficiently large. Similarly, $\lim_{x\to-\infty} f(x) = L$ means that $f(x)$ can be made as close to L as we please by taking the value of x sufficiently small (in the negative direction).

Example 2: $\lim_{x\to\infty}(1/x) = 0$. This is simply the observation that by taking x sufficiently large we can make its reciprocal as close to zero as we please. Similary $\lim_{x\to-\infty}(1/x) = 0$.



The line $y = 0$ is approached by the graph of $y = 1/x$ as $x \to \infty$ and also as $x \to -\infty$. It is called a *horizontal asymptote* of the graph. Also the vertical line $x = 0$ is approached by the graph as $x \to 0$. It is called a *vertical asymptote* of the graph. Finding such horizontal and vertical asymptotes for a graph aids in sketching the graph.

Example 3: $\lim_{x\to\infty} \frac{x}{x^3+2} = 0$. This limit is often said to be of the "$\frac{\infty}{\infty}$" form because both the numerator and denominator approach ∞ as x increases without bound. It is not immediately evident what the quotient does as $x \to \infty$, so we try to rewrite it in a more transparent way. Dividing the numerator and denominator by the highest power of x in the denominator often helps:

$$\lim_{x\to\infty} \frac{x}{x^3+2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{1+\frac{2}{x^3}} = \frac{0}{1} = 0$$

After dividing the numerator and denominator by x^3 the resulting expression is again a quotient, but this time not of the "$\frac{\infty}{\infty}$" form. Both the numerator and denominator have finite limits, so we are able to evaluate the limit of the quotient as the quotient of the limits when both exist.

Example 4: Evaluate the limit $\lim_{x\to\infty}(x^4 - x^2 + 2)/(x^3 + 3)$. Again this is an "$\frac{\infty}{\infty}$" form, so we try dividing numerator and denominator by the hightest power of x in the denominator:

$$\lim_{x\to\infty} \frac{x^4 - x^2 + 2}{x^3 + 3} = \lim_{x\to\infty} \frac{x - \frac{1}{x} + \frac{2}{x^2}}{1 + \frac{3}{x^3}} = \infty$$

In the rewritten expression it is clear that the numerator approaches ∞ while the denominator approaches 1. The quotient therefore increases without bound.

Example 5: Evaluate the limit $\lim_{x\to\infty}(5x^3+6x^2+x+1)/(3x^3+x^2-x+7)$. Dividing numerator and denominator by the highest power in the denominator, we have

$$\lim_{x\to\infty}\frac{5x^3+6x^2+x+1}{3x^3+x^2-x+7}=\lim_{x\to\infty}\frac{5+\frac{6}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{3+\frac{1}{x}-\frac{1}{x^2}+\frac{7}{x^3}}=\frac{5}{3}$$

The examples involve rational functions $R(x)=P(x)/Q(x)$, i.e. quotients of polynomials. Three typical situations are illustrated. In Example 3 the degree of the numerator is less than the degree of the denominator, and the limit is 0. In Example 4 the degree of the numerator is greater than the degree of the denominator, and the limit is ∞. In Example 5 the degrees of the numerator and denominator are the same, and the limit is the quotient of the coefficients of the highest power terms. These three cases are often codified as rules:

Dominant Term Rule: For the limit $\lim_{x\to\infty}P(x)/Q(x)$, where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m,

1. If $n<m$, the limit is 0,
2. If $n>m$, the limit is $\pm\infty$,
3. If $n=m$, the limit is the quotient of the coefficients of the highest powers.

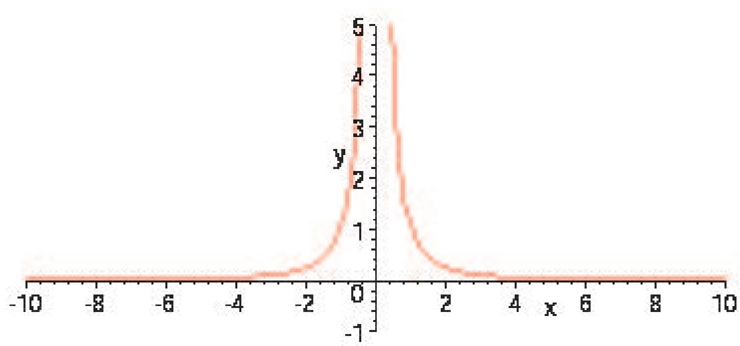
Our advice is to ignore this rule as just so much clutter. Memorizing more rules just obscures the technique illustrated in the three examples. The technique applies to more than just limits of rational functions, hence warrants your attention.

Example 6: As an example involving a non-rational function, evaluate $\lim_{x\to\infty}x/\sqrt{3x^2+2}$. In this example, thinking in the dominant term style, we suspect that the denominator will behave very much like the function $\sqrt{3x^2}=\sqrt{3}x$. Thus we guess that the limit is $1/\sqrt{3}$. This is indeed the case as the following computation shows:

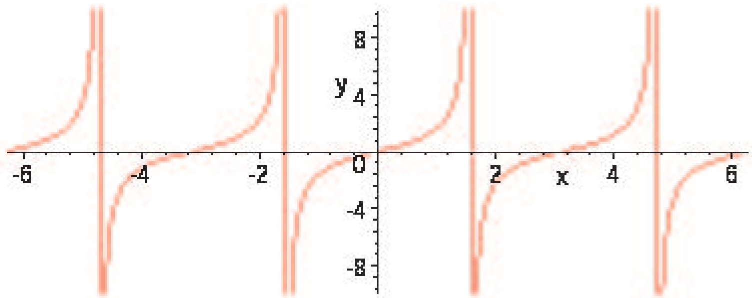
$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{3x^2+2}} &= \lim_{x\to\infty}\frac{x}{\sqrt{x^2(3+\frac{2}{x^2})}}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{3+\frac{2}{x^2}}}=\frac{1}{\sqrt{3}}\end{aligned}$$

Note that the *Dominant Term Rule* does not apply directly to this example, but the technique underlying it does. Before concluding this section, we give a few examples of infinite limits:

Example 7: Evaluate $\lim_{x\to0}1/x^2$. The limit does not exist, of course, since it is of the form "$\frac{1}{0}$". But let us analyse the right-hand and left-hand limits at 0. Clearly $\lim_{x\to0^+}(1/x^2)=\infty$ as does the left-hand limit (the function is an even function). In this case the right-hand and left-hand limits do not differ, so we can also write $\lim_{x\to0}(1/x^2)=\infty$. Although the limit DNE, the notation signals additional information about how the function $1/x^2$ behaves in the vicinity of 0. The y-axis is a vertical asymptote, and the x-axis is a horizontal asymptote.



Example 8: $\lim_{x\to\pi/2}\tan x=\lim_{x\to\pi/2}\frac{\sin x}{\cos x}$ does not exist (it is of the form "$\frac{1}{0}$"). In this case $\lim_{x\to\pi/2^+}\tan x=-\infty$ and $\lim_{x\to\pi/2^-}\tan x=\infty$ (see the graph below). The lines $x=\pi/2+n\pi$, n any integer, are vertical asymptotes.



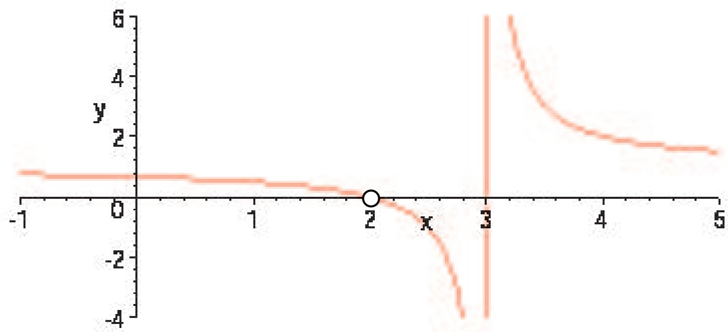
Example 9: Find the horizontal and vertical asymptotes, if any, of

$$f(x) = \frac{x-2}{(x^2-5x+6)} + 1, \quad (x \neq 2, x \neq 3)$$

and then sketch the graph. We note, first, that the denominator factors; thus

$$f(x) = \frac{x-2}{(x^2-5x+6)} + 1 = \frac{x-2}{(x-2)(x-3)} + 1 = \frac{1}{x-3} + 1 = \frac{x-2}{x-3}$$

From the last expression we see that the line $x = 3$ is a vertical asymptote and that the right-hand limit is ∞ and the left-hand limit is $-\infty$. We notice, also, that the function $(x-2)/(x-3)$ vanishes at $x = 2$. Thus the graph appears to cross the x-axis at $x = 2$ except that this point is not in the domain of the original function. The point is missing from the graph.



Summary: In this section we extended our notion of limit to include points $x = a$ where the limit does not exist but where we could use the notation $\pm\infty$ to provide additional information about the behavior of the function. We also extended the limit definition to include limits as $x \to \pm\infty$. With limits thus defined, and with our skill honed in computing some of them, we now turn in the next sections to the several ideas of calculus that depend on the notion of limit—continuity, derivative, and integral.

Applet: Limits of Functions **Try it!**

Exercises: Problems **Check what you have learned!**

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