Limits of Functions

The notion of *instantaneous velocity* was defined in terms of a *limit* of average velocities over shorter and shorter time intervals. For a particle moving on the x-axis, with position $x(t)$ at time t, we introduced the notation

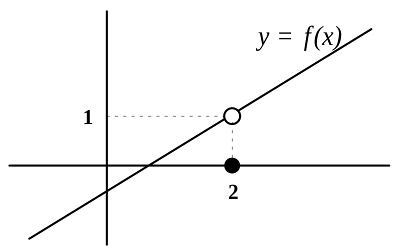
$$v(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}$$

to formalize this process. In our first computations we proceeded informally in evaluating several limits, confident that our intuition would not lead us astray. Indeed, that is not a bad approach. Newton operated informally throughout his work, revolutionizing the understanding of dynamic motion. It was not until nearly 150 years later that it was found necessary to revisit the foundations of calculus and to provide precise definitions for its underlying notions such as limit.

Definition 1: We say that a function f approaches the limit L as x approaches a, written $\lim_{x\to a} f(x) = L$, if we can make $f(x)$ as close to L as we please by taking x sufficiently close to a.

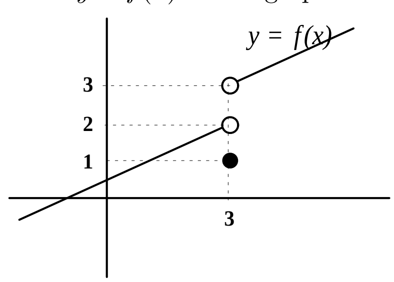
We hasten to note that this "definition" serves to introduce notation and language for talking about limits, not to resolve the deeper questions of what we mean by phrases such as "close to L" or what is meant by "sufficiently close". Suffice it to say that precise meanings can be given to these phrases, and that we may safely follow our intuitions in interpreting and working with them. From time to time we will point out some of the subtleties as necessary.

Example 1: Let f be the function whose graph is shown. The values of f lie on a straight line except that f has a different value at $x = 2$.



The graph is not continuous at $x = 2$ because a single point has been "moved" to $(2, 0)$. Notice that $\lim_{x\to 2} = 1$, the value that $f(x)$ is "trying to achieve" at $x = 2$, although its actual value is $f(2) = 0$. The fact that the *limit* of f and the *value* of f differ is the essence of the "discontinuity of f" at that point.

Example 2: Again, consider the function $y = f(x)$ whose graph is shown.



In this case the limit of f as $x \to 3$ does not exist. As the point $x = 3$ is approached from the right the function "tries to achieve" the value 3. From the left it "tries to achieve" the value 2. It is customary to introduce the notion of *right-hand limit* and *left-hand limit* in such cases and to write $\lim_{x\to 3^+} f(x) = 3$ and $\lim_{x\to 3^-} f(x) = 2$. The actual value of $f(3)$ is irrelevant to the consideration of whether a limit does or does not exist. In fact $f(3) = 1$, differing from both the right-hand limit and the left-hand limit. Again, the function is "discontinuous" at $x = 3$.

Theorem 1: The limit of f as $x \to a$ exists if and only if both the right-hand and left-hand limits exist and have the same value. I.e.

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L$$

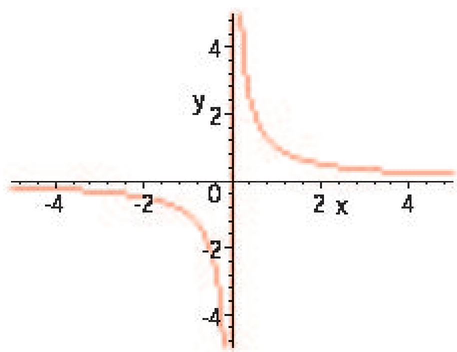
Example 3: $\lim_{x\to 1}\frac{x-1}{x+1}=\frac{0}{2}=0$. In this case the function has the value 0 at the point $x=1$, and we may thus calculate the limit by simply evaluating the function. In doing this we are implicitly using the fact that the function is *continuous* at the point. I.e. the value that the function is "trying to achieve" at the point coincides with its actual value at the point. This characterization of "continuity" is made precise in the next sections.

Example 4: Calculate the limit $\lim_{x\to 2}(x^2-4)/(x-2)$. Here we cannot substitute the value $x=2$ into the expression, for this would yield the meaningless form "0/0". Our approach is to simplify the expression so as to make its behavior more obvious:

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\\ &= \lim_{x\to 2}(x+2)=4\end{aligned}$$

What we have done through algebraic simplification is reveal the "offending" factor $x-2$ that let to the "0/0" form. After cancelling it we can evaluate the limit by substituting $x=2$. The technique is simple. Its justification is a bit deeper. The cancellation of the factor $x-2$ is legitimate only when $x\neq 2$ since division by zero is not defined. Thus the two functions $(x^2-4)/(x-2)$ and $x+2$ are equal for all values of x *except* for $x=2$, and this means that they have the same limit at $x=2$. But the function $x+2$ is *continuous* at $x=2$ (as we will see in the following sections) and so its limit as $x\to 2$ agrees with its value there. The mystery of using the value of a function when computing its limit is thus resolved.

Example 5: Consider $\lim_{x\to 0}(1/x)$. Here the limit does not exist (DNE). The values of the function increase without bound as x approaches 0 from the right (we say that $\frac{1}{x}\to\infty$), and they decrease without bound as x approaches 0 from the left (we say that $\frac{1}{x}\to-\infty$). In particular they do not approach any real number L.



We will write $\lim_{x\to 0^+}(1/x)=\infty$ in this case even though the right-hand limit DNE (∞ is not a real number). Similarly we write $\lim_{x\to 0^-}(1/x)=-\infty$.

Example 6: Let $f(x)=(x+3)/(x-5)$ and consider $\lim_{x\to 5}f(x)$. The analysis of this example is essentially the same as that of Example 5. From the right we have $\lim_{x\to 5^+}f(x)=+\infty$ and from the left $\lim_{x\to 5^-}f(x)=-\infty$. Thus the given limit DNE.

Many of the examples above involved calculation of $\lim_{x\to a}f(x)$ where $f(x)$ is a rational function, i.e. a quotient of two polynomials. Our general line of attack for such problems was first to try substituting $x=a$. If this yields a meaningful real value L, then the limit is L. On the other hand if it yields one of the meaningless forms "0/0" or "$a/0$" ($a\neq 0$) we must look further. In the "$a/0$" case the limit does not exist nor do the right-hand or left-hand limits. But it still may be possible to classify these limits as $\pm\infty$. In the "0/0" case the limit may or may not exist. We proceed by rewriting the expression algebraically, cancelling the factor $x-a$ if possible, and then study the resulting simpler expression.

Theorem 2: If $\lim_{x\to a}f(x)=A$ and $\lim_{x\to a}g(x)=B$ both exist, then

1. $\lim_{x\to a}(f(x)+g(x))=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=A+B$
2. $\lim_{x\to a}(f(x)-g(x))=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)=A-B$
3. $\lim_{x\to a}(f(x)g(x))=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=A\cdot B$

4. $\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x) = A/B \ \ (B \neq 0)$

Example 7: Find $\lim_{x\to 1}(x^2+3x-1)/(x^3+4x+6)$. First we apply part 4 of Theorem 2, which states that the limit of a quotient is the quotient of the limits if each limit exists. Then we apply parts 1 and 3:

$$\begin{aligned} \lim_{x\to 1}\frac{x^2+3x-1}{x^3+4x+6} &= \frac{\lim_{x\to 1}(x^2+3x-1)}{\lim_{x\to 1}(x^3+4x+6)} && \text{(Thm.2, Part 4)} \\ &= \frac{1^2+3\cdot 1-1}{1^3+4\cdot 1+6} && \text{(Thm.2, Parts 1 and 3)} \\ &= \frac{3}{11} \end{aligned}$$

In this case the substitution of $x = 1$ is completely meaningful, yielding the value of the limit. (We remark again that this is implicitly using the continuity of the rational function at the point $x = 1$.)

Example 8: Consider $\lim_{x\to 0}\frac{|x|}{x}$. Since $|x|$ is defined piecewise ($|x| = x,\ x \geq 0$, and $|x| = -x,\ x < 0$), we consider separately the right-hand and left-hand limits. For $x > 0$ we have

$$\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = 1.$$

And for $x < 0$ we have

$$\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\frac{-x}{x} = -1.$$

Since the right-hand and left-hand limits have different values, the limit does not exist.

Example 9: Let $f(x) = |x-1|/(x^2-1)$ and consider $\lim_{x\to 1} f(x)$. Again we consider the cases $x < 1$ and $x > 1$ separately. For $x < 1$

$$\lim_{x\to 1^-}\frac{|x-1|}{x^2-1} = \lim_{x\to 1^-}\frac{-(x-1)}{x^2-1} = \lim_{x\to 1^-}\frac{-1}{x+1} = -\frac{1}{2},$$

and for $x > 1$

$$\lim_{x\to 1^-}\frac{|x-1|}{x^2-1} = \lim_{x\to 1^-}\frac{x-1}{x^2-1} = \lim_{x\to 1^-}\frac{1}{x+1} = \frac{1}{2},$$

Since the right-hand and left-hand limits have different values, the limit DNE.

Example 10: Let $f(x) = 1/x$. Let us compute $\lim_{h\to 0}[f(x+h)-f(x)]/h$.

$$\begin{aligned} \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{-1}{x(x+h)} \\ &= -\frac{1}{x^2} \end{aligned}$$

The notation $\lim_{x\to a} f(x) = L$ means that L is a real number and that the values of $f(x)$ approach L arbitrarily closely as the values of x approach a. One way to make this precise would be to invent a game: if you name a positive number ϵ, no matter how small, a second player Dr. Delta will try to respond with a positive number δ that is small enough to guarantee that $|f(x)-L| < \epsilon$ whenever $0 < |x-a| < \delta$. If Dr. Delta has a winning strategy for this game, i.e. if he can win the game no matter how cleverly (how small) you choose your number ϵ, then the limit of $f(x)$ as $x \to a$ exists and is L.

The game described above is the famous (perhaps infamous) δ–ϵ definition of limit. Let's examine an example of the game in operation:

Example 11: Let $f(x) = x^2$ and let $L = 4$. Let us "prove" that $\lim_{x\to 2} f(x) = L$. Playing the game, suppose you name the positive number $\epsilon = 0.0001$. Dr. Delta will respond by naming $\delta = 0.000033$. Aha, he wins this game. For whenever $|x - 2| < 0.000033$ we see that

$$\begin{aligned} |f(x) - L| &= |x^2 - 4| = |x - 2| \cdot |x + 2| && (1) \\ &< 0.000033 \cdot 2.000033 = 0.000066001089 < 0.00001. && (2) \end{aligned}$$

Sorry, he's gotcha! But you might then come back with a still smaller number $\epsilon = 10^{-100}$. No problem! Dr. Delta will just respond by choosing δ to be one-third of whatever number you name. For if $\delta = \frac{1}{3}\epsilon$, then whenever $|x - 2| < \delta$ we see that

$$\begin{aligned} |f(x) - L| &= |x^2 - 4| = |x - 2| \cdot |x + 2| && (3) \\ &< \frac{1}{3}\epsilon \cdot (2 + \frac{1}{3}\epsilon) < \frac{1}{3}\epsilon \cdot 3 = \epsilon && (4) \end{aligned}$$

In computing the inequalities we used the fact that $|2 + \frac{1}{3}\epsilon| < 3$. This is certainly true if your value of ϵ is small. If you were impetuous enough to choose a large value ($\epsilon > 1/3$, for example) you would just be making Dr. Delta's life easier. What the example shows is that he has a winning strategy for playing the game. He just gives the function $\delta = \frac{1}{3}\epsilon$, and then he can go home and leave you to play the game by yourself. This means that Dr. Delta has "proved" that $\lim_{x\to a} f(x) = L$.

Ooh! Having done that example we will never do it again, at least not in this book. Newton and Leibniz, the co-inventors of calculus, never gave such proofs. Suffice it to say that should the need ever arise we could drag the δ–ϵ argument from the shelf to settle a difficult limit case. But for the large part we will rely upon our intuition about limits, and this will enable us to concentrate on the techniques of calculus that will give us so much power in solving problems.

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