The Legacy of Galileo, Newton, and Leibniz

In the space age we take motion for granted—tracking earth-orbiting satellites, delivering supplies to the emerging international space station from a space shuttle, understanding the dynamics of a dying, collapsing star that is about to become a supernova, or even looking back in time to the birth of our universe in the "big bang". All of these phenomena relate to gravity and the wonderful way it governs our world. It boggles the minds of 21st century humans to remember that only 400 years ago the greatest minds among humans were only beginning to achieve a rudimentary understanding of gravity.

Galileo was interested in falling bodies. It is said that he spent his pasttime atop the leaning tower of Pisa, releasing objects to study how they fall to the ground. In that day it was widely believed that heavy objects fall to the ground faster than light objects. Rather than design experiments to test such a belief, however, the prevailing approach was to look to the classical Greek scientists for answers. It took a Galileo to move beyond the authority of the ancients and to forge a new scientific methodology—*observe nature, construct experiments to test what you observe, and construct theories that explain the observations.* This is the essence of the *scientific method* that has revolutionized our understanding of the natural world since Galileo's time.

Galileo learned from his experiments with falling objects that, neglecting the resistance of the atmosphere, objects of different weights fall at the same rate. Indeed he quantified this principle by discovering that the distance a body drops in time t is proportional to t^2. He was not able to explain why such a simple mathematical law should apply to falling bodies, however. It took Isaac Newton and the methods of calculus to do that. Gottfried Leibniz, co-inventor of calculus, took a slightly different point of view but also studied rates of change in a general setting.

In modern language we would say that an object, falling under the influence of gravity, will have constant *acceleration* of $9.8m/sec^2$, that consequently the *velocity* of the object is a linear function of time t and the distance the body has fallen is a quadratic function of t. Newton was able, using his new tools of calculus, to explain why falling bodies behave in this way. His *laws of motion* and of *universal gravitation* drew under one simple mathematical theory Newton's laws of falling bodies, Kepler's laws of planetary motion, the motion of a simple pendulum, and virtually every other instance of dynamic motion observed in the universe. The explosion of scientific knowledge of the past 400 years is due largely to the direction set by Galileo, Newton, and Leibniz.

Newton looked at Galileo's work on falling bodies and generalized the questions to any moving object. His question: How do we *find* the velocity of a moving object at time t? Newton was astute enough to realize that the more fundamental question was "What in fact do we mean by *velocity* of the object at the instant of time t"? We know how to find the *average velocity* of an object during a time interval $[t_1, t_2]$:

Definition 1: The average velocity during a time interval is the distance traveled divided by the elapsed time, i.e.

$$\text{Average velocity over } [t_1, t_2] = \frac{\text{distance traveled}}{t_2 - t_1}$$

Example 1: A car traveling from Boston to Hanover in three hours, a distance of 135 miles, had an *average velocity* of $135/3 = 45$ miles per hour. The traffic officer that stopped the car in the wilds of New Hampshire and issued a speeding ticket (92 miles per hour in a 65 mile per hour zone) was not thinking of average velocity, however. He said "you were traveling 92 miles per hour at the *instant* my radar gun caught you". The police officer was of the same mind as Isaac Newton in believing one can talk about the *instantaneous velocity* at a time t.

In fact the velocity measured by the officer was, in reality, also an average velocity, albeit measured over a very short time interval. The elapsed time in this case was the few milliseconds between two pulses from the radar gun, and the distance traveled was the few centimeters the car moved during this time. Although it thus begs the question of *instantaneous* velocity, it contains the seeds of a precise definition. Namely, to find the *instantaneous* velocity of an object at time t one can obtain, instead, the *average* velocity during a very short time interval about time t. As the length of the time interval gets shorter and shorter the average velocity *approaches* the instantaneous velocity. In short, the instantaneous velocity is the limiting value of average velocities computed over shorter and shorter time intervals. Making the notion of limit precise is the business of calculus.

Example 2: Suppose that an object is moving vertically so that its position at time t is given by $y = 4.9t^2$. Then the average velocity during the time interval $[1, 2]$ is

$$\text{ave vel on } [1,2] = \frac{y(2) - y(1)}{2 - 1} = \frac{4.9(2)^2 - 4.9(1)^2}{1} = 14.7 \text{ m/sec}$$

What is the (instantaneous) velocity at time $t = 1$? Taking a "short" time interval $[1, 1+h]$ we compute the average velocity

$$\begin{aligned}
\text{ave vel on } [1, 1+h] &= \frac{(4.9)(1+h)^2 - (4.9)(1)}{(1+h) - 1} \\
&= \frac{4.9(1 + 2h + h^2 - 1)}{h} \\
&= 4.9\frac{2h + h^2}{h} \\
&= 4.9(2 + h), \;\; h \neq 0
\end{aligned}$$

Letting $h \to 0$ the last expression for average velocity evidently approaches the value $2(4.9) = 9.8$ m/sec. In the next section we will formalize the process of taking limits, but for the moment our method is to simplify the given expression algebraically until the limiting value is clear. Note that the first three expressions, above, are meaningless ("0/0") when we substitute $h = 0$. The last expression, however, does have a value when $h = 0$, and this is also the value of the limit that we seek. The key to this is the notion of *continuity* which we take up in the next section. [Notice, by the way, that the limiting value is the same whether we approach the point $y = 1$ from the right or from the left, i.e. whether we allow h to approach 0 through positive or negative values.]

Definition 2: Let $x(t)$ be a function that gives the position at time t of an object moving on the x-axis. Then

$$\text{Ave vel}[t_1, t_2] = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

$$\text{Velocity}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

The quotient in the last expression is called a *difference quotient*.

Example 3: Suppose a particle moves along the x-axis with its position at time t given by $x(t) = t^2 - 4t + 1$. Initially the particle was at the point $x(0) = 1$. Find its average velocities during the intervals $[1, 2]$, $[2, 3]$, and $[1, 3]$. And find its (instantaneous) velocity $v(t)$ at time t.

$$\text{Ave vel}[1,2] = \frac{(2^2 - 4 \cdot 2 + 1) - (1 - 4 + 1)}{2 - 1} = \frac{-3 - (-2)}{1} = -1$$

$$\text{Ave vel}[2,3] = \frac{(3^2 - 4 \cdot 3 + 1) - (2^2 - 4 \cdot 2 + 1)}{3 - 2} = \frac{-2 - (-3)}{1} = 1$$

$$\text{Ave vel}[1,3] = \frac{(3^2 - 4 \cdot 3 + 1) - (1 - 4 + 1)}{3 - 1} = \frac{-2 - (-2)}{2} = 0$$

$$\begin{aligned}
v(t) &= \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \\
&= \lim_{h \to 0} \frac{[(t+h)^2 - 4(t+h) + 1] - [t^2 - 4t + 1]}{h} \\
&= \lim_{h \to 0} \frac{t^2 + 2th + h^2 - 4t - 4h + 1 - t^2 + 4t - 1}{h} \\
&= \lim_{h \to 0} \frac{2ht + h^2 - 4h}{h} = \lim_{h \to 0} (2t + h - 4) = 2t - 4
\end{aligned}$$

Having found a general formula for $v(t)$ we can analyze the motion of the particle quite easily. At the initial time $t = 0$ the velocity was $v(0) = -4$, meaning that the particle was moving to the left with a speed of 4 units per second, and its position was $x(0) = 1$. It continued moving to the left until $t = 2$ at which time its velocity was $v(2) = 0$ and its position was $x(2) = -3$. From that time onward its velocity is positive so it is moving to the right.

Summary: In this section we began with the common notion of *average velocity*, the ratio of distance traveled to elapsed time, and moved to a definition of *instantaneous velocity*. The latter depended on the notion of "taking a limit". Newton and Leibniz were among the first to recognize the fundamental importance of limits and to develop methods for working with them. The collection of such techniques is what has become known as *calculus*, and it is the reason that Newton and Leibniz are remembered as the co-inventors of calculus. In the subsequent sections we will retrace their steps.

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