Modeling Rates of Change: Introduction to the Issues

One of the great breakthroughs of the seventeenth century was an understanding of motion. According to Aristotle, the force due to gravity affects the speed of an object, so that heavy objects fall faster than lighter ones. This belief was not challenged until the late sixteenth and early seventeenth centuries, when Galileo set out to establish through experiments the true effect of gravity on objects in free fall. The now famous story of Galileo dropping balls off the Leaning Tower of Pisa is probably apocryphal. We do know, however, that he conducted many experiments that involved him rolling balls down an incline plane, a ramp, where it is easier to measure speeds because the ball can be slowed down.

Suppose now that we turn to modern methods, and consider an object dropped vertically from rest near the surface of the earth. We can imagine, for example, that we drop a ball from the edge of a cliff. As the object falls, we record in a table, the distances it covers for a period of time. Here is such a table:

time (s)	distance (m)
0.10	0.049
0.20	0.196
0.30	0.441
0.40	0.784
0.50	1.225
0.60	1.764
0.70	2.401
0.80	3.136
0.90	3.969
1.00	4.900

The times are measured in seconds, and the distances in meters. How much can we learn about the distance function from this table of values? We faced similar questions in the first section of the book, and our approach there was to seek an elementary function that would model the data. In this section we are going to take a different tack by adopting the point of view of Newton, and study the speed and acceleration of the object. That is, we are going to use the characteristics of motion that are familiar to us all, namely, average speed and acceleration, to try and get information about the distance function.

0.1 Average Speed

If we are going to use the speed and acceleration of the falling object to get information about the distance it falls as a function of time, then a reasonable first step is to add two columns to the above table, one for speed and the other for acceleration. However, we immediately encounter a problem: How do we measure the speed at a given time? For example, if we drive from Philadelphia to New York City, a distance of 90 miles, in an hour and a half, then our average speed is $90/1.5 = 60$ miles per hour. But this tells us very little about the speed we are going when we cross the state line into New York. This instantaneous speed is what a policeman's radar gun purports to measure. We will return to this issue later and see if indeed that is the case. For now, let us try to do the best we can with what we can calculate from the data, namely, average speeds.

An average speed is always computed over an interval of time. In fact, **average speed** is defined to be **change in distance divided by change in time**. Returning to the table of distances, where the times are recorded 0.1 seconds apart, we will add a column for the speeds. As an approximation to the instantaneous speed, the speed at time t will be the average speed over the interval $[t, t+0.1]$. (Even though we have chosen the interval of time to the right of t, we could just as easily have chosen the interval to the left.) We will also add a column of accelerations by deriving it from the column of speeds. That is, we will calculate the acceleration at t as the speed at $t+.1$ minus the speed at t, divided by 0.1; thus, it is the average acceleration over the interval $[t, t+.1]$ where we are using the average speeds to approximate the instantaneous speeds.

Derived Table of Speeds and Accelerations $[t, t+0.1]$			
time t (s)	distance (m)	speed (m/s)	acc (m/s/s)
.100000	.049000	1.470000	9.800000
.200000	.196000	2.450000	9.800000
.300000	.441000	3.430000	9.800000
.400000	.784000	4.410000	9.800000
.500000	1.225000	5.390000	9.800000
.600000	1.764000	6.370000	9.800000
.700000	2.401000	7.350000	9.800000
.800000	3.136000	8.330000	9.800000
.900000	3.969000	9.310000	
1.000000	4.900000		

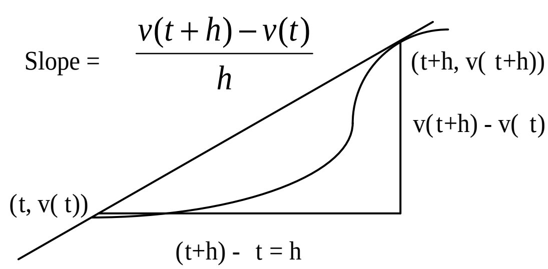
Before we investigate the implications of the derived table, let's go back to our everyday experiences with automobile travel. Speed is how fast we are traveling. It is the rate of change of distance with respect to time. Acceleration is the rate of change of speed with respect to time. We accelerate when we pull out to pass a car, and we decelerate when we approach a stop sign. When we accelerate, we feel our bodies being pushed against the seatback. The snap of the head we feel when we suddenly push down hard on the gas pedal is due to "jerk." Jerk is the rate of change of acceleration.

Now, we are ready to consider the derived table. Setting aside the fact that we are using averages as approximations, our approach has been to calculate the rate of change of distance, namely, the speed, and then the rate of change of speed, or acceleration. If we tried to go further and use the table to calculate the rate of change of acceleration, or jerk, we would not get very far: the accelerations are all constant. So, the jerk is zero.

Applet: Average Velocity **Try it!**

0.2 The Meaning of Constant Acceleration

Suppose the speed of a falling object is given by the function $v(t)$. Then the average acceleration over the interval $[t, t+h]$ is given by the quotient $\frac{v(t+h)-v(t)}{h}$; in the derived table, h = 0.1. Here is an important observation that connects this physical definition with algebra and geometry: geometrically, the quotient, which we will refer to as a *difference quotient*, is the slope of the line through the points $(t, v(t))$ and $(t+h, v(t+h))$ on the graph of the function v.



Thus, we can articulate the following question that is motivated by the constant accelerations in the derived table. Suppose we are given a finite number of points whose adjacent x-coordinates are h units apart. Suppose that the slope of the line through any two successive points is always the same; that is, these slopes are constant. Then how can we characterize the points geometrically?

From our work with the elementary functions, we should realize that all of the points must lie on a (straight) line. Hence, we have arrived at an important realization about what the derived table tells us. Assuming that the average speeds give good approximations to the instantaneous speeds, we can conclude that the speed function is linear.

0.3 Hypotheses and Open Questions

Let's take stock of where we are. We began with a table of values for the distance of a falling object as a function of time. Motivated by a desire to study this function by taking into account what we know about the characteristics of motion, we produced the derived table from calculations. These calculations of speed and acceleration in actuality were average and not intantaneous values. Even so, from the fact that the derived accelerations turned out to be constant, we have arrived at two hypotheses:

1. The acceleration of a falling object is constant as a function of time.
2. The speed of a falling object is linear as a function of time.

There are also at least two open questions that our work has raised about falling objects:

1. Can we find a description (i.e., a formula) for the distance function? This is really the question we started with.
2. How can we get better approximations to the instantaneous speeds?

Let's address the second question first. We have recorded the distances in our table 0.1 seconds apart, and have used the average speed over the interval $[t, t + 0.1]$ as an approximation to the speed at time t. Clearly, we could do better if we recorded the speeds 0.01 seconds apart, or even 0.001 seconds apart, etc. We can describe this process as *looking for a limiting value* of the average speeds as the interval h between successive times shrinks to zero. So, we should make a note for ourselves and return to this point later: For a given time t, it makes sense to define the instantaneous speed (or velocity) at t by way of this process. Symbolically, we write:

$$v(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h}$$

For the moment, let's return to the radar gun that we mentioned earlier. In reality, a radar gun does not measure the exact speed at an instant of time. Instead, it attempts to carry out the above limiting process by calculating the average speed of a moving car for a very small value of h, namely, the interval of time associated with one pulse of the radar gun. The limiting process that we have described symbolically demands more. That is, we want h, though non-zero, to get smaller and smaller without end; and we take the limiting value, if it exists, of the difference quotients (the average speeds) to be the instantaneous velocity. We will discuss this approach in more detail in the next section.

Now, for the first question. Given our hypotheses, we can find a representation for the distance function.

0.4 The Distance Function is Quadratic

We are assuming that the speed (or velocity) is linear, and that the initial speed is 0. Thus, the velocity has the form $v(t) = at$, for some constant a. We can derive the formula for distance from this assumption and two other facts. The first is a formal statement of what we already know, and the second is one that could have been established in Galileo's laboratory:

1. In the case of constant velocity, distance equals velocity times time.
2. In the interval of time from 0 to t, in the case of linear velocity the distance traveled is the same as if the object had traveled at a constant velocity equal to one-half the final velocity.

Now, putting all of this information together, we have

$$s(t) = \frac{v(t)}{2} \cdot t = \frac{at}{2} \cdot t = \frac{at^2}{2}$$

That is, the distance function is quadratic. Also, from the first line of the table we can find the value of a: $a(.1^2)/2 = .049$ implies that $a = 9.8$ meters per second per second.

Note that a comparison with the table shows this to be the (constant) acceleration of the object, due to gravity no less.

0.5 Average Rate of Change

So far, we have started with a table of values, and generated a derived table. By studying the derived table, we have been able to learn a great deal of information about the underlying function that corresponds to the original table. Using the derived table has been very productive in this case.

Let's now work in the other direction. We will begin with an elementary function and then generate a derived table for it to see if anything important stands out. To compute the first derived, we need an analogous notion of average speed.

Definition: Given a function f, the *average rate of change* of f over an interval $[x, x+h]$ is

$$\frac{f(x+h) - f(x)}{h}$$

That is, if $y = f(x)$, the average rate of change of f is the change in f divided by the change in x over the interval. (The change in f is $f(x+h) - f(x)$, and the change in x is $(x+h) - x = h$.) The average rate of change is also what we have called the *difference quotient* over the interval.

We will use the formula for the average rate of change of f over an interval $[x, x+h]$ to compute the first derived. The second derived will be the change in the first derived divided by h over that interval. We will use a small value of h, the distance between successive x-values, because we have agreed that this gives a better approximation to the instantaneous rate of change at a point. Note that in analogy with what we did in the case of motion, we are defining the *instantaneous rate of change* of a function at a point x to be the limit of the average rates of change over intervals $[x, x+h]$ as $h \to 0$. Let's carry out this process for the function $y = e^x$.

Derived Table for $y = e^x$

x	e^x	1st derived	2nd derived
.001000	1.001000	1.001501	1.003000
.002000	1.002002	1.002504	1.002000
.003000	1.003005	1.003506	1.004000
.004000	1.004008	1.004510	1.005000
.005000	1.005013	1.005515	1.006000
.006000	1.006018	1.006521	1.008000
.007000	1.007025	1.007529	1.007000
.008000	1.008032	1.008536	1.009000
.009000	1.009041	1.009545	1.010000
.010000	1.010050	1.010555	1.012000
.011000	1.011061	1.011567	1.011000
.012000	1.012072	1.012578	1.014000
.013000	1.013085	1.013592	1.014000
.014000	1.014098	1.014606	1.014000
.015000	1.015113	1.015620	1.017000
.016000	1.016129	1.016637	1.017000
.017000	1.017145	1.017654	1.019000
.018000	1.018163	1.018673	1.018000
.019000	1.019182	1.019691	
.020000	1.020201		

The table is interesting indeed. A comparison of the columns for e^x and the 1st derived function would lead us to believe that the derived function of $y = e^x$ is $y = e^x$; that is, both the derived function and its derived function (the 2nd derived function) are the same as the original function. We can experiment further with this hypothesis by decreasing the value of h and observing the outcome. The same conclusion will pertain: It appears that the rate of change of the function $y = e^x$ at a point is the value of the function at that point.

Modeling Note: Suppose we did not know that the derived table came from the function $y = e^x$. If we examined the first two columns from the left, we might think that the underlying function was $1 + x$. The next column, 1^{st} Derived, would tell us we were wrong, however, because even data resulting from measurements should be close to 1. Thus, we would discard $1 + x$ as a possibility. If we persisted with polynomials and tested $1 + x + \frac{1}{2}x^2$, then the columns for the function and the 1^{st} Derived would match fairly well but the 2^{nd} Derived column would not (again, it should be close to 1). But the polynomial $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ would give close enough results for all the columns of the table that we would not know how to decide upon the underlying function. The only point here is that over short intervals polynomials can give good approximations to more complicated functions. In particular, linear, quadratic , and cubic functions serve well for this purpose. We will touch on this topic again when we have developed additional analytical tools, and we will also explain how we got the coefficients of the polynomials.

The Exponential Function $y = e^x$ is Special Indeed

Above, we concluded from the table that the rate of change of the function $y = e^x$ at a point $x = a$ is e^a. This certainly seems special, doesn't it? We have used the word *special* before in connection with $y = e^x$. When we studied its properties, we found that it was special among all exponential functions because it has a slope of 1 at the origin. In our current investigation, this begs the question: What is the rate of change of $y = e^x$ at $x = 0$? Based on our work above, we know how to answer this question. At $x = 0$, we have to investigate the behavior of the difference quotients, the average rates of change,

$$\frac{e^{0+h} - e^0}{h}$$

for small values of h.

Difference Quotient for $y = e^x$ at $x = 0$

h	difference quotient
.100000000000000	1.051709180756480
.010000000000000	1.005016708416800
.001000000000000	1.000500166708000
.000100000000000	1.000050001670000
.000010000000000	1.000005000000000
.000001000000000	1.000000500000000
.000000100000000	1.000000050000000
.000000010000000	1.000000000000000
.000000001000000	1.000000000000000
.000000000100000	1.000000000000000

From the table, we are led to believe that this limit is 1. That is,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

This means that the limit of the difference quotients at $x = 0$ equals the slope of the tangent line to the graph of $y = e^x$ at $x = 0$. Coincidence? Certainly not. In fact, we will see that, for any point x, the instantaneous rate of change of a function at the point (what we have been calling the limit of the average rates of change, or of the difference quotients, at the point) is equal to the slope of the tangent line to the graph of the function at the point. And even more generally, this holds for each of the elementary functions. Truly amazing.

Applet: Rate of Change **Try it!**

0.6 Our Agenda for This Chapter

Our attempt to find an explicit formula for the distance function of a falling object has raised many issues that require further investigation and work. On the other hand, our findings so far point to the high probability of success if we proceed along the following lines. In fact, this list will form our agenda for the rest of the chapter.

1. Develop a more explicit notion of limit.
2. Adopt the definition of *instantaneous rate of change at a point* as the limit of the average rates of change, or difference quotients, at the point as the length of the interval approaches zero.
3. Explore the geometric meaning of the definition of instantaneous rate of change at a point.
4. Apply the definition to each of the elementary functions to see if there are formula-like rules for calculating the instantaneous rate of change.
5. Use the definition of instantaneous rate of change and its consequences to obtain explicit functions for the position, velocity, and acceleration of a falling object.

We began this section with data recorded in a table, and used the data to calculate some related physical quantities. On the basis of our findings, we developed hypotheses and conjectures. Given that these hypotheses are true, we proved some mathematical results. The outcomes have contributed much to our understanding of the original data. Now, we have the task of going back and filling in the deatils in a systematic way, making definitions explicit, verifying the hypotheses, and producing rock-solid mathematical results. The above list is a blueprint for carrying out this task. Our study of the particular example of the falling object leads us to believe that there is a general theory lurking in the background. We will soon see that this indeed is the case, and that the theory will turn out to be *differential calculus*.

Applet: Derived Function **Try it!**

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**