

Exponential and Logarithm Functions

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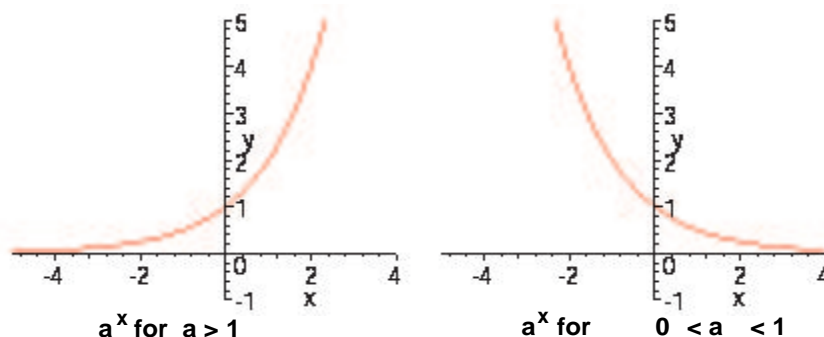
Throughout the preceding sections we have talked about the useful class of *elementary functions* but have not yet defined precisely what we mean by that term. We have cited polynomial functions, rational functions, and trigonometric functions as belonging to the class. And we have emphasized for each of these the unique properties that make them valuable in modeling real world problems. For example the polynomial and rational functions are the basis of nearly every algebraic expression we write. And the trigonometric functions are *periodic*, making them the functions of choice in modeling physical processes that repeat themselves.

In this section we add the *exponential functions* and *logarithmic functions* to the list. You will be happy to know that that is close to the end of the story. From the basic functions named above we obtain all the remaining elementary functions by applying arithmetical operations—addition, subtraction, multiplication, and division—and by using composition of functions and inverses of functions. Much of calculus is involved in studying the properties of the basic functions named above and in learning to use them in applications that rely upon their unique properties. The full class of elementary functions provides a rich source from which we can draw functions to represent a variety of real world objects and model their behavior.

Exponential functions make their most dramatic debut in population modeling. The population of Mexico, for example, increased at the rate of 2.6% per year during the 1980's. Beginning with a population of 67.38 million people in 1980, the population increased each year by a factor of 1.026. Thus in 1981 the population was $67.38(1.026)$ million, in 1982 it was $67.38(1.026)^2$ million, and in general it is $P(t) = 67.38(1.026)^t$ where t is the number of years that have elapsed since 1980. This is an *exponential function*. It is called an exponential function because the base is constant, in this case the constant 1.026, and the independent variable t is in the exponent. The base represents the *growth factor* by which the population increases each year. Notice that the growth *rate* of 2.6% corresponds to a growth *factor* of 1.026. A growth *rate* of $r\%$ per year corresponds to a growth *factor* of $1 + r/100$.

In general, for any positive constant a , we have an exponential function a^t . The key property of such functions is the constant ratio a between the values of the function in any two consecutive years.

Definition 1: Let a be a positive real number. Then $P(x) = Ba^x$ is called a general exponential function.



Because $a^0 = 1$, B is the value of the exponential function at 0: $P(0) = Ba^0 = B$.

When the growth factor $a > 1$ the graph is increasing. From our hint that exponential functions model population growth we would expect that. What is surprising is the rate at which they grow. Even exponential functions that begin their lives by growing slowly eventually go through the roof with gusto. This has disastrous implications for the world if constant growth rates are sustained. The term *exponential growth* refers to exponential functions with positive growth rates (growth factors greater than 1).

When the growth factor is less than 1 the graph is decreasing. This might model the balance in your bank account where as a result of inflation the value of your nest egg decreases each year. If the inflation rate is 3.5%, the growth rate of the value of your balance is -3.5% per year, corresponding to a growth factor of $1 + (-.035) = 1 - .035 = .965$. Clearly we should be referring instead to the “shrinkage rate” and the “shrinkage factor”. The shape of the second curve above will explain why inflation can give you such a

pain in your liver. The term *exponential decay* is an apt expression of the behavior of exponential functions with negative growth rates (growth factors between 0 and 1).

True Confessions We have behaved in our discussion as though we know what we mean by a^x . In fact we have never given a complete definition. If x is an integer or a rational number we *have* defined the value of a^x . For example $a^0 = 1$. For a positive integer n the definition is $a^n = \underbrace{a \cdot a \cdot a \cdots a}_n$. For a positive

rational number $r = m/n$ we define $a^r = \sqrt[n]{a^m}$. And for a negative integer or rational number we define $a^r = 1/a^{-r}$.

These are our definitions from the treatment of exponents in algebra. And we know, in addition, certain basic laws of exponents:

Laws of Exponents

If $a > 0$ and $b > 0$, and x and y are any real numbers, then

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| (i) $a^0 = 1$ | (ii) $a^{x+y} = a^x a^y$ |
| (iii) $a^{-x} = \frac{1}{a^x}$ | (iv) $a^{x-y} = \frac{a^x}{a^y}$ |
| (v) $(a^x)^y = a^{xy}$ | (vi) $(ab)^x = a^x b^x$ |

But we have not defined a^x when x is not a rational number. For example we have not defined the quantity 2^π nor $3^{\sqrt{2}}$. We can hardly consider a^x to be a continuous function if it remains undefined for irrational numbers. In effect its graph is full of holes since the irrational numbers are densely mixed with the rational numbers.

Having confessed we will now sidestep this issue, important as it is, with the promise that it will be fixed later on when we have additional tools of calculus available. In the meantime we will “fill in the holes” by requiring that a^x be a continuous function. This would imply that the quantity $3^{\sqrt{2}}$ is defined, for example, (relying on the fact that $\sqrt{2}$ is the infinite non-repeating decimal 1.4142135623730950488...) as the limit of the sequence of values

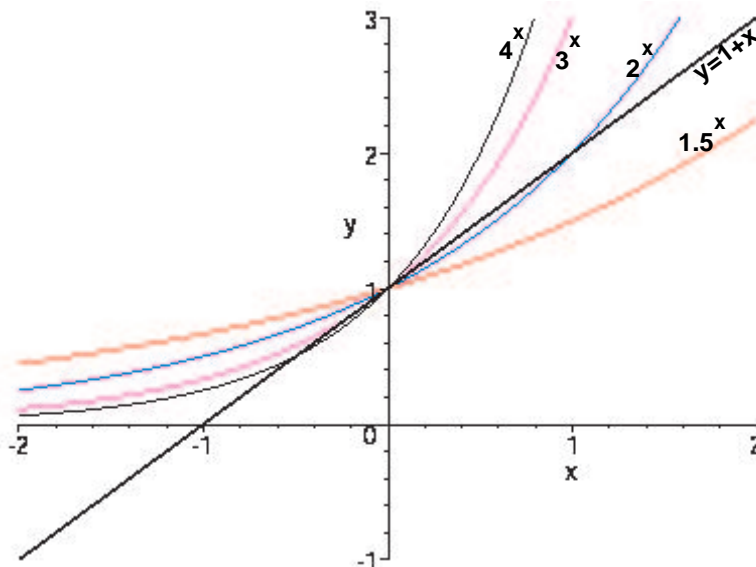
$$3^{1.4}, 3^{1.41}, 3^{1.414}, 3^{1.4142}, 3^{1.41421}, \dots$$

That such limits always exist needs proof, of course, and that will come. Until then we will, without embarrassment, assume that the exponential function a^x is defined and continuous for all real values of x and that all the usual laws of exponents hold. This will enable us to move on to the applications that make these functions so important.

Example 1: We can use the laws of exponents to ease our task when computing with exponentials. For example $2^{10} = (2^5)^2 = 32^2 = 1024$. And $2^{20} = (2^{10})^2 = 1024^2 = 1,048,576$.

Example 2: We can freely interchange exponential and “root” notation. For example $\sqrt{x} = x^{\frac{1}{2}}$, $x^{\frac{3}{5}} = \sqrt[5]{x^3}$.

Example 3: The graphs of functions a^x for different values of $a > 1$ are quite similar.



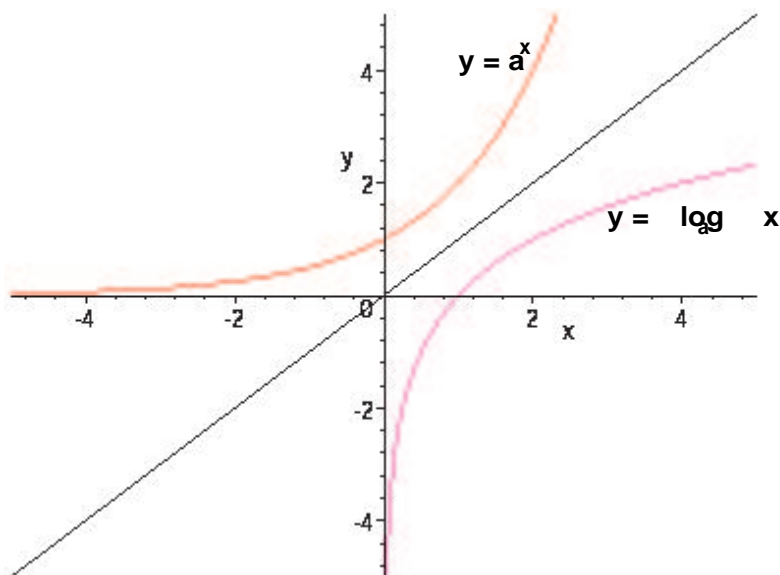
They differ in their slope at the point $(0, 1)$. (We are using the word *slope* to mean the slope of the tangent line at the point on the curve. We will go beyond the everyday usage we rely on here and make the notion precise in Chapter 3. Being able to do so constitutes one of the triumphs of calculus.) We notice that 1.5^x and 2^x have slopes less than 1 (the graph of $y = 1 + x$ was included in the plot for comparison), while 3^x and 4^x have slopes greater than 1. Apparently the slope at $(0, 1)$ increases as a increases. Being curious, we can ask whether there is a value of a for which the slope of a^x at $(0, 1)$ is exactly equal to 1? We expect such a value between 2 and 3, and indeed there is! We will show that there is a number $e = 2.718281828459045\dots$, an infinite non-repeating decimal (and hence irrational) number, for which e^x has slope at $(0, 1)$ which is exactly 1. This will turn out to be the most “natural” base for an exponential function and will become the standard for calculus.

Applet: Comparing Exponential Functions Try it!

The Inverse of a^x : If $a > 1$ the function a^x is increasing on $(-\infty, \infty)$, and if $0 < a < 1$ it is decreasing on this interval. In either case it is a 1:1 function and so has an inverse.

Definition 2: The inverse of the general exponential function a^x , written as $\log_a x$, is called the *general logarithm function*. It is defined by the relations

$$y = a^x \Leftrightarrow x = \log_a y.$$



Many properties of $\log_a x$ follow immediately. Its graph is the reflection in the line $y = x$ of the graph of a^x . Its domain is $(0, \infty)$ since that is the range of a^x . Its range is $(-\infty, \infty)$ since that is the domain of a^x . The value of $\log_a 1 = 0$ since $a^0 = 1$. And the characteristic laws of logarithms hold:

Laws of Logarithms

If $a > 0$, $b > 0$, $a \neq 1$, and $b \neq 1$, then

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|--|---|
| (i) $\log_a 1 = 0$ | (ii) $\log_a xy = \log_a x + \log_a y$ |
| (iii) $\log_a \frac{1}{x} = -\log_a x$ | (iv) $\log_a \frac{x}{y} = \log_a x - \log_a y$ |
| (v) $\log_a x^y = y \log_a x$ | (vi) $\log_a x = \frac{\log_b x}{\log_b a}$ |

All of these laws follow from the laws of exponents. We prove (ii) and (vi) as examples. For (ii), let $\log_a x = u$ and $\log_a y = v$. Then $\log_a x + \log_a y = u + v$. But $a^u = x$ and $a^v = y$, so $xy = a^u a^v = a^{u+v}$. Thus, finally,

$\log_a xy = u + v = \log_a x + \log_a y$. For (vi), let $\log_a x = u$. Then

$$\begin{aligned} a^u = x &\Leftrightarrow \log_b a^u = \log_b x \\ &\Leftrightarrow u \log_b a = \log_b x \quad (\text{using (v)}) \\ &\Leftrightarrow u = \frac{\log_b x}{\log_b a} \end{aligned}$$

The Number e: We have already pointed to the number $e \approx 2.718281828\dots$ that plays a special role for exponential functions. Namely the exponential function e^x crosses the y-axis at the point $(0, 1)$ with slope exactly equal to 1. For the moment we are taking that as our definition of e . (As promised, we will be returning later to give a precise definition of e^x as a continuous and differentiable function.)

Definition 3: The *natural exponential function* e^x is that exponential function that crosses the y-axis with slope 1. Its inverse $\log_e x$ is called the *natural logarithm function* and is denoted more simply by $\ln x$.

The two functions e^x and $\ln x$ are the ones that occupy prime space in calculus. We will see that all other exponential and logarithm functions can be expressed in terms of these two, hence in a sense are redundant. We will also learn why they are termed “natural”. It has to do with the fact that they have the simplest differentiation formulas among all exponential and logarithm functions. With e^x and $\ln x$ added to our repertoire of basic functions, we have also completed our definition of the class of *Elementary Functions* of calculus.

Of course e^x and $\ln x$, as special cases of the general exponential and logarithm functions, satisfy all the laws for exponents and logarithms listed above. But they are central enough in calculus that we state them again here.

Properties of e^x

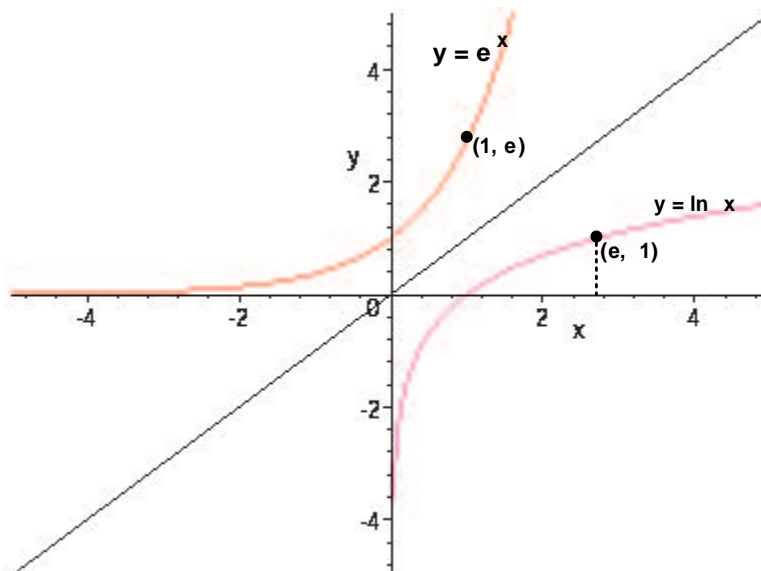
Domain of e^x is $(-\infty, \infty)$, and its Range is $(0, \infty)$

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|--------------------------------|--|
| (i) $e^0 = 1$ | (ii) $e^{x+y} = e^x e^y$ |
| (iii) $e^{-x} = \frac{1}{e^x}$ | (iv) $e^{x-y} = \frac{e^x}{e^y}$ |
| (v) $(e^x)^y = e^{xy}$ | (vi) $a^x = e^{x \ln a}$, ($a > 0$) |

Properties of $\ln x$

Domain of $\ln x$ is $(0, \infty)$, and its Range is $(-\infty, \infty)$

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|----------------------------------|--|
| (i) $\ln 1 = 0$ | (ii) $\ln xy = \ln x + \ln y$ |
| (iii) $\ln \frac{1}{x} = -\ln x$ | (iv) $\ln \frac{x}{y} = \ln x - \ln y$ |
| (v) $\ln x^y = y \ln x$ | (vi) $\log_a x = \frac{\ln x}{\ln a}$ |



Exercises: [Problems](#) Check what you have learned!

Videos: [Tutorial Solutions](#) See problems worked out!