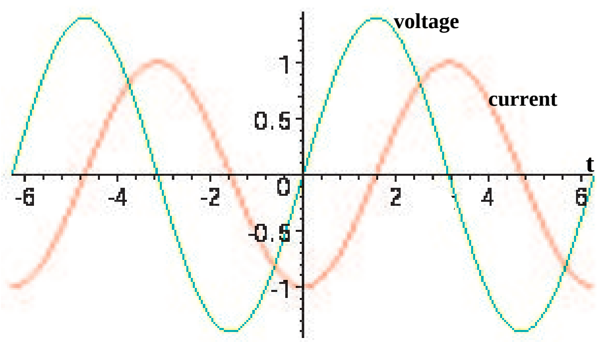
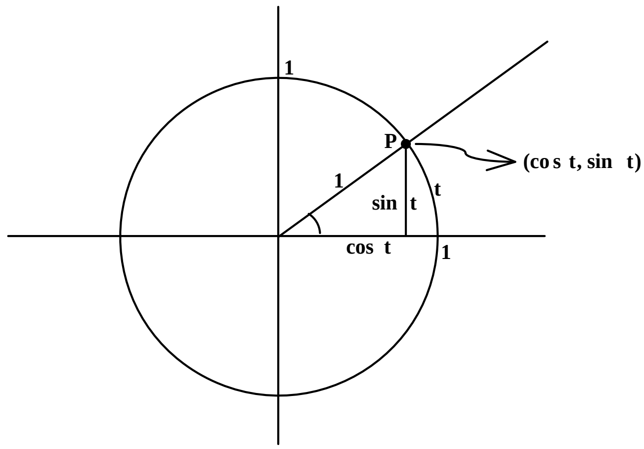
Trigonometric Functions

Modeling with Trigonometric Functions: You first met the trigonometric functions in algebra and trigonometry in high school. In a typical trigonometry course the functions $\sin x$, $\cos x$, and the other related functions $\tan x$, $\sec x$, etc., are defined as ratios of sides in right triangles. The focus is on measuring the sides and angles of triangles, hence the term *trigonometric functions.* Much of your attention was directed to applications in geometry growing out of this connection with right triangles as well as to identities that express relationships between the several trigonometric functions. In calculus the focus changes. The trigonometric functions are defined in terms of arclength on a unit circle, and the emphasis is on the periodic behavior of the trigonometric functions. It is their periodicity that leads to their most important applications in science—modeling phenomena that repeat as a function of time. Simple harmonic motion (sinusoidal motion), light and sound waves, electricity, gravitational waves in the universe, oscillations of the pendulum of a clock, oscillations of atomic crystals on which our most accurate time keeping is based—all these are periodic phenomena that are modeled mathematically by the trigonometric functions. The essential characteristics of any periodic motion are the *amplitude* (how big are the oscillations?) and *period* (how long is a single wave?). Sometimes the *phase* (how much has the wave been translated to the right or delayed in time?) is also important, as when one is comparing two related wave forms such as voltage and current—in the standard way of generating electricity the current lags the voltage by 90 degrees (the phase angle is $\pi/2$. Their graphs might look as follows:



Here the amplitude of the voltage is 1.4, the amplitude of the current is 1, both have period 2π, and the current lags the voltage by a phase angle of $\pi/2$. The *period* tells us how long a single wave is, and, especially when the independent variable is time, this is sometimes described instead in terms of *frequency* (how many oscillations occur in a unit distance or unit of time).

Definition 1: The Trigonometric Functions: In calculus we define the trigonometric functions in terms of arc length on a unit circle. Choose a point P on the circle (see the plot below), at a distance t from the positive x-axis measured counterclockwise along the circle. Then the trigonometric functions $\cos t$ and $\sin t$ are defined to be the coordinates of the point P.



The arclength t in the plot is a measure of the central angle that subtends the arc. It is called the *radian* measure of the angle. An angle of $360°$ subtends the entire circle whose length is 2π, thus $360° = 2\pi$ radians, and $1° = \pi/180$ radians. Also $180° = \pi$ radians, $90° = \pi/2$ radians, and $60° = \pi/3$ radians We will consistently use radian measure of angles in calculus.

Reference to the figure also tells us that our new definition of the trigonometric functions is consistent with the usual definitions as ratios of sides of a right triangle. In the small right triangle in the figure with angle t (radians), the hypotenuse of the triangle is 1, the adjacent side has length $\cos t$, and the opposite side has length $\sin t$. Thus, for example, the ratio of the adjacent side to the hypotenuse is $\frac{\cos t}{1}$, as it should be. The main thing to remember is that we are measuring angles in radians instead of degrees, so $\cos t$ and $\sin t$ are functions of the *real variable* t. There is thus no longer any reason to restrict our attention to angles less than $360°$. We may measure any positive distance along the unit circle that we wish, and since the point P on the circle returns to its starting point when $t = 2\pi$, we note that the values of $\sin t$ and $\cos t$ repeat when we "wrap around" the circle, i.e. when t increases by a multiple of 2π. This gives us the fundamental *periodic* behavior of $\sin t$ and $\cos t$, the property that makes them so useful in modeling periodic phenomena.

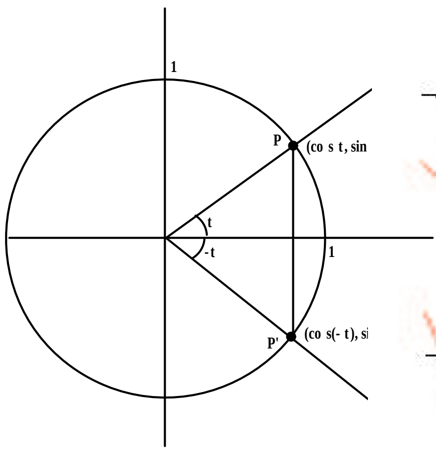
We can also allow t to take on negative values. These simply correspond to distances measured *clockwise* along the circle, beginning at the point $(1, 0)$. The sin and cos functions thus have domain $-\infty < t < \infty$.

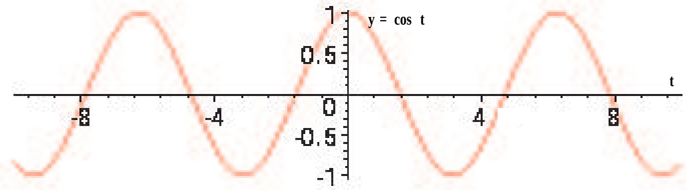
Applet: Definitions of sin(x)and cos(x) **Try it!**

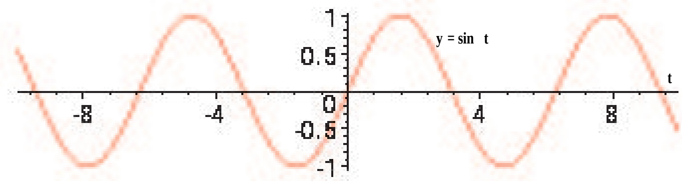
Theorem 1: The trigonometric functions sin and cos are defined for all real values of t, and are *periodic with period* 2π. I.e. they satisfy

$$\sin(t + n \cdot 2\pi) = \sin t \text{ for any real } t \text{ and any integer } n.$$

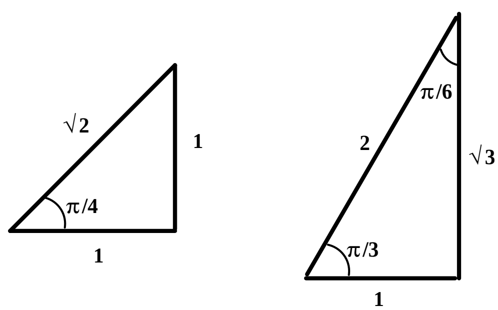
Many of the familiar properties of the trigonometric functions follow immediately from their definition. Since $(\sin t, \cos t)$ is a point on the unit circle it is clear that $\sin^2 t + \cos^2 t = 1$ for all values of t. (Note: we follow the usual convention of writing $\sin^2 t$ instead of the more cumbersome $(\sin t)^2$; however, when doing calculations, we do type the latter so that a computer can understand it.) Moreover it is also clear that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$ (see the figure below), thus cos is an *even* function and sin is an *odd* function. These symmetries and the periodicity, along with special values such as $\cos 0 = 1$, $\cos \pi/2 = 0$, $\cos \pi = -1$, $\sin 0 = 0$, $\sin \pi/2 = 1$, etc., enable us to sketch their graphs:



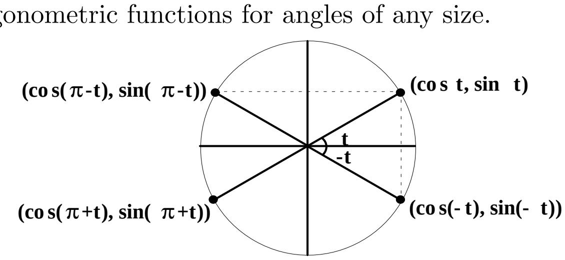


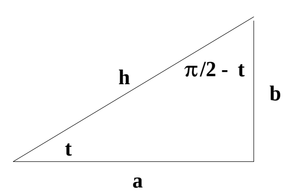


Other special values of sin and cos can be read directly from the geometry of the unit circle. For example the angle $\pi/4$ (or $45°$) determines an isosceles right triangle with hypotenuse 1; thus the legs of this triangle both have length $1/\sqrt{2}$. It follows that $\cos\pi/4 = \sin\pi/4 = 1/\sqrt{2}$. In making such computations it is often more convenient to use a reference triangle that is *similar* to the small one in the unit circle but with more convenient dimensions. In this example it is easier to refer to a triangle with legs of length 1 and hypotenuse of length $\sqrt{2}$ and to read off the values of the trigonometric functions using their traditional definitions as ratios of sides of the triangle. Reference triangles for several special angles are shown below. From these can you read off, for example, the values of $\sin\pi/6$, $\cos\pi/6$, $\sin\pi/3$, and $\cos\pi/3$?



The examples above dealt with angles in the first quadrant ($0 \le t \le \pi/2$). Angles in other quadrants can be handled using, again, the geometry of the unit circle. An angle t in the second quadrant, for example, can be "reflected" in the y-axis and compared with the angle $\pi - t$ in the first quadrant. Examining the figure below it is clear that $\cos(\pi - t) = -\cos t$ and $\sin(\pi - t) = \sin t$. In particular, therefore, $\cos(5\pi/6)$ =-$cos(\pi/6)$, and the value can then be read off from one of the reference triangles shown above. Angles in the third quadrant can be reflected in the origin and compared with an angle in the first quadrant. And angles in the fourth quadrant can be reflected in the x-axis. By considering only angles in the first quadrant, therefore, and using the reference triangle method for computing sin and cos of special angles, we can compute the values of these trigonometric functions for angles of any size.





We should mention, finally, the identities $\cos(\pi/2 - t) = \sin t$ and $\sin(\pi/2 - t) = \cos t$. The angles t and $\pi/2 - t$ are *complementary.* They "share" a single right triangle. Do you see that the ratio $\frac{a}{h}$ is $\cos t$ as well as $\sin(\pi/2 - t)$? As a general rule, it is not necessary to remember hundreds of unrelated facts and identities for the trigonometric functions. A little understanding of the geometry involved goes a long way.

Definition: Other trigonometric functions: For the record we define the other trigonometric functions that are often used. They are

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$$

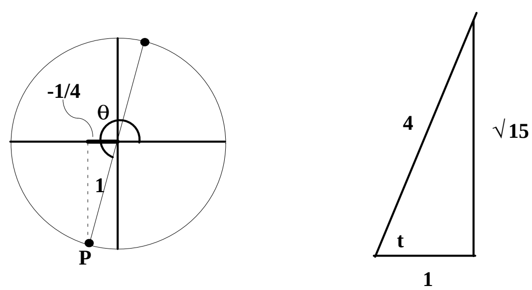
$$\sec t = \frac{1}{\cos t} \qquad \csc t = \frac{1}{\sin t}$$

Notice that all of these additional trigonometric functions are defined in terms of sin and cos. And, in fact, the sine function also can be defined in terms of cos since, for angles in the first quadrant

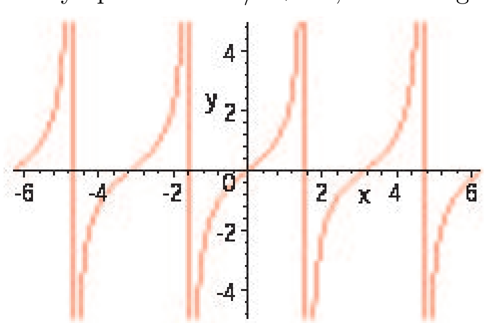
$$\sin t = \sqrt{1 - \cos^2 t}.$$

In a sense, then, most of the trigonometric functions are superfluous—only one of them is needed, and all the others can be defined in terms of that one. But this would obscure the rich set of identities relating the different trigonometric functions. It is this algebraic richness that can be exploited in solving many problems. We do not intend to pause here for an extensive survey of such identities, nor do we recommend memorizing endless lists of identities and properties. Rather, we have mentioned a few basic identities that follow more or less directly from the unit circle definition, and we prefer to "derive" more identities only as the need for them arises.

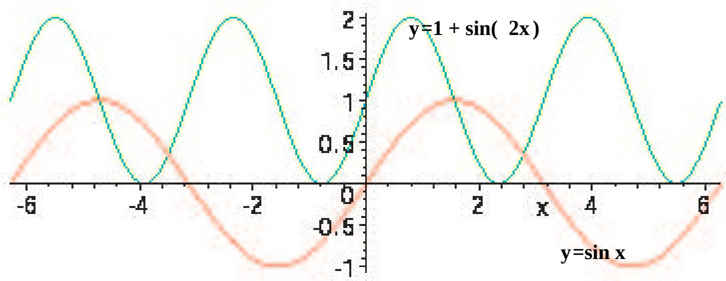
Example 1: Suppose θ is a third quadrant angle and $\cos\theta = -1/4$. Find $\tan\theta$. Refer to the figure below. We first observe that the sign of $\tan\theta$ is positive since both sin and cos are negative in the third quadrant. Then, reflecting the point P in the origin, we may use the more convenient reference triangle shown in the figure to compute the value of $\tan\theta$. Thus $\tan\theta = +\tan t = \sqrt{15}$. The values of the other trigonometric functions can also be computed from this reference triangle, taking into account their signs in the third quadrant. So $\sin\theta = -\sqrt{15}/4$, $\sec\theta = -4$, etc..



Example 2: Sketch the graph of $\tan x$. Since $\tan x = \frac{\sin x}{\cos x}$, we notice that the x-intercepts of the graph occur where $\sin x = 0$, i.e. at the points $x = n\pi$ where n is an integer. Moreover $\tan x$ is undefined whenever $\cos x = 0$. Thus the graph has vertical asymptotes $x = \pi/2 + n\pi$, n an integer.



Example 3: Sketch the graph of $f(x) = 1 + \sin 2x$. We can visualize this graph as the graph of $\sin x$ compressed by the factor 2 (all horizontal distances multiplied by 1/2) and shifted up 1 unit. Moreover since $\sin x$ has period 2π, the function $f(x)$ has period π. We show the graphs of both functions below.



Applet: Trigonometric Identities **Try it!**
Exercises: Problems **Check what you have learned!**
Videos: Tutorial Solutions **See problems worked out!**