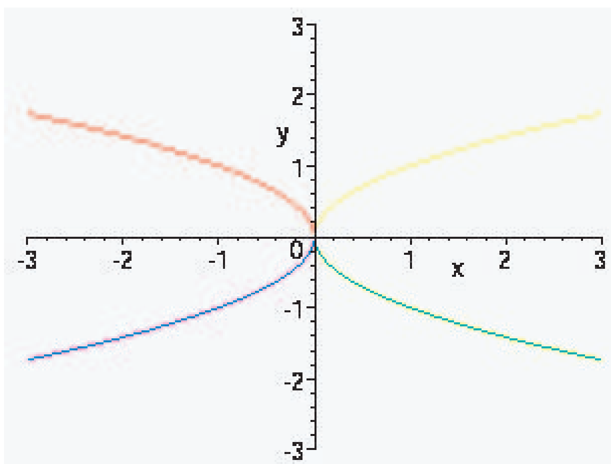
Defining New Functions from Old

The notion of symmetry discussed in the previous section can be used to define new functions from old. For example the graphs of $y = \sqrt{-x}$ and $y = \sqrt{x}$ are mirror images of each other in the y-axis. Each has a domain that is only half of the real axis, hence cannot be an even function. However, replacing x by $-x$ in the latter has the effect of reflecting its graph in the y-axis, turning it into the graph of the former. More generally:

Theorem 1: Reflections in special lines: For an equation in x and y

1. Replacing x by $-x$ corresponds to reflecting the graph of the equation in the y-axis.
2. Replacing y by $-y$ corresponds to reflecting the graph of the equation in the x-axis.
3. Replacing both x and y by their negatives corresponds to reflecting the graph of the equation in the origin.
4. Interchanging x and y in an equation corresponds to reflecting the graph of the equation in the line $y = x$.

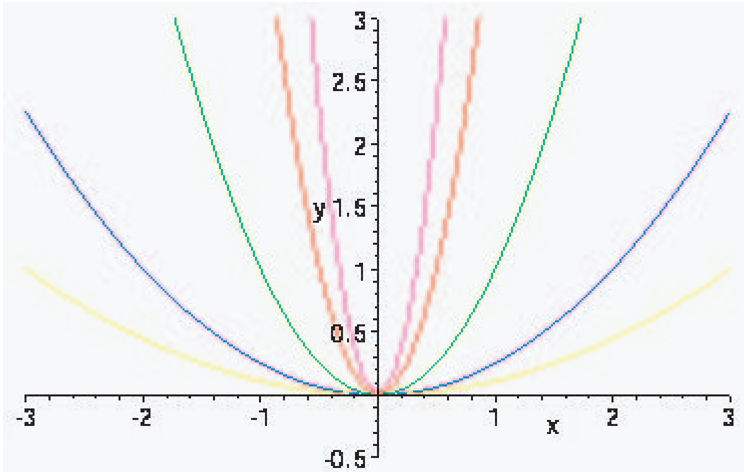
Example 1: The plot below shows the graphs of four functions: $\sqrt{x}$, $\sqrt{-x}$, $-\sqrt{x}$, and $-\sqrt{-x}$. Identify in the plot the graph of each of these functions by plotting a few points for each. To get started, verify that $\sqrt{x}$ is in the upper right-hand quadrant, and $-\sqrt{x}$ is in the lower right-hand quadrant.



Even though we often can define new functions from existing ones using symmetry, acquiring a sufficient repertoire of functions for solving all the problems that can arise may seem hopeless. Modeling real-life and scientific situations places strenuous demands upon our mathematical experience and knowledge. As we mentioned in Section 1.1, however, a small number of basic functions, most of which already occur in your precalculus study, can serve as *building-blocks* for a much wider and very useful class of functions, the *Elementary Functions*. It is on those building-blocks that we focus in this section. And it is with the class of elementary functions under our belts that we will begin our study of calculus.

In Section 1.2 we already noted some of the ways in which new functions can be defined from old. The function $f(x) = \sqrt{-x}$, for example, is closely related to the function $g(x) = \sqrt{x}$. We can think of it as obtained by reflecting (the graph of) $g(x)$ in the y-axis.

Scaling a graph: Another simple geometric transformation that yields new functions from old is a *stretch*—either parallel to the x-axis or parallel to the y-axis. For example compare the graphs of $y = x^2$ and $y = (cx)^2$, for various values of the constant c. When x is replaced by cx, horizontal distances are multiplied by the factor $1/c$. When $0 < c < 1$ this is seen geometrically as "stretching" the graph of x^2 parallel to the x-axis. And when $c > 1$ it appears as a "compression" of the graph.

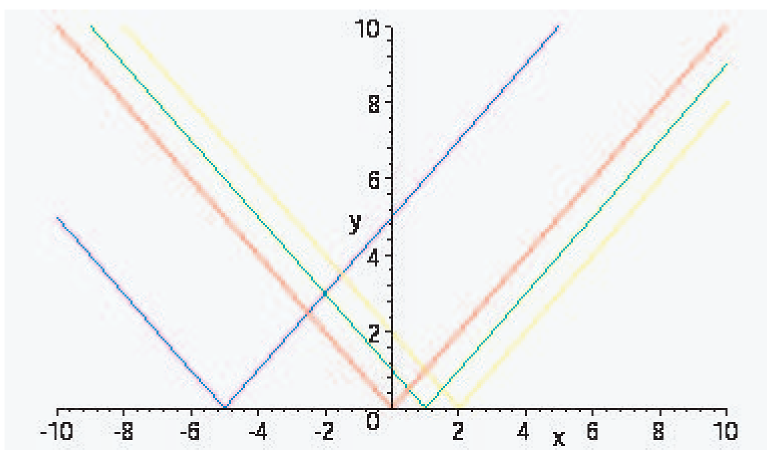


Identify in this plot the graphs of $(\frac{1}{3}x)^2$, $(\frac{1}{2}x)^2$, x^2, $(2x)^2$, and $(3x)^2$. Notice that we only need to be familiar with the one basic function x^2 in order to "understand" the other functions $(cx)^2$.

Theorem 2: Replacing x by cx in a function $y = f(x)$ results in a horizontal stretching or compression of the graph of f. When $0 < c < 1$ the graph is *elongated* horizontally by the factor $1/c$, and when $c > 1$ it is *compressed* horizontally by the factor $1/c$.

Applet: Stretching Graphs **Try it!**

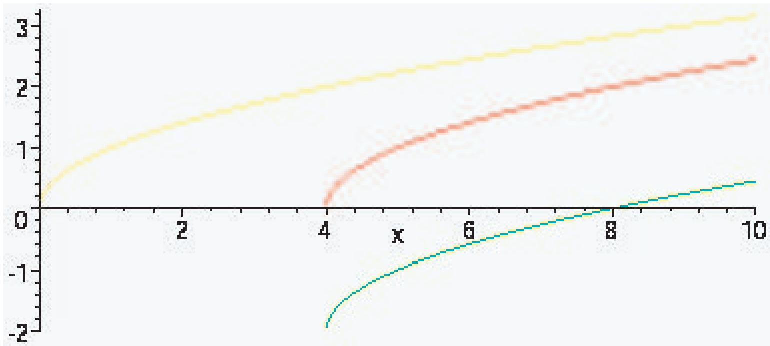
Shifting a graph: The graph of a function may be shifted a units horizontally by replacing x by $x - a$. In the following plot, for example, the graphs of $|x|$, $|x-1|$, $|x-2|$, and $|x+5|$ are shown. Notice that replacing x by $x-2$ shifts the graph of $|x|$ 2 units to the *right*. And replacing x by $x+5$ shifts it 5 units to the *left*.



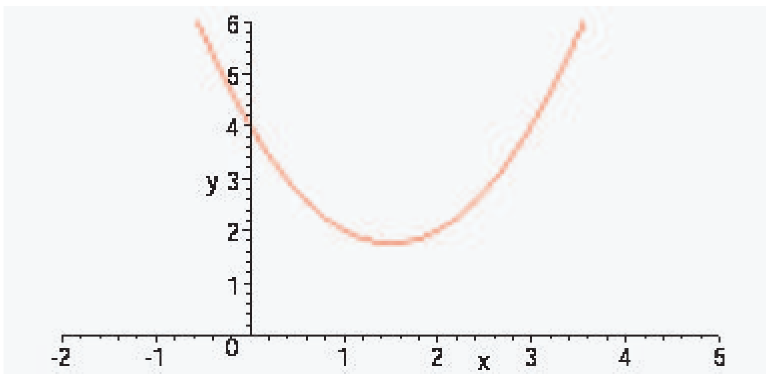
Theorem 3: Assume that the constant a is positive. To shift the graph of a function $f(x)$ to the right by a units, replace x by $x - a$. To shift it to the left by a units, replace x by $x + a$.

Vertical stretching and shifting may also be accomplished. The graph of $cf(x)$ is obtained by elongating the graph of $f(x)$ by the factor c in a direction parallel to the y-axis. And the graph of $f(x)+a$ is obtained by shifting the graph of $f(x)$ upward a units. We will not state these two additional geometrical transformations as separate theorems, but they will often be applied in graphing elementary functions.

Example 2: Draw the graph of the function $y = \sqrt{x-4} - 2$. We compare this function with $\sqrt{x}$. Replacing x by $x - 4$ shifts the graph 4 units to the right. Then subtracting 2 shifts the graph 2 units *down*.



Example 3: Draw the graph of $y = x^2 - 3x + 4$. Completing the square in this quadratic expression, we write the equation in the form $y = (x - \frac{3}{2})^2 + \frac{7}{4}$. We then recognize the graph as that of x^2, shifted 3/2 units to the right and 7/4 units up.



Applet: Shifting Graphs **Try it!**
Applet: New Functions from Old **Try it!**
Applet: New Functions from Old Game **Try it!**

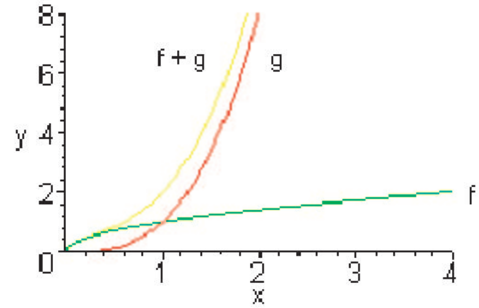
Arithmetical operations and Composition: Another way to build new functions from old is implicit in our use of algebraic expressions to define functions. The polynomial $P(x) = x^3 + 5x$ for example is constructed from the constant function 5 and the identity function x using addition and multiplication. (From 5 and x we obtain $5x$ by multiplication. $x^3 = x \cdot x \cdot x$ is also obtained from x by multiplication. And $P(x)$ is just the sum of these two functions. Similarly the rational function $R(x) = \frac{x^3+5x}{3x^2+1}$ is obtained from the constant functions 1, 3, 5, and the identity function x using division along with addition and multiplication. Finally, the function $f(x) = \sqrt{x^3 + 5x}$ is obtained by *composition* from the polynomial $P(x)$ and the square root function $\sqrt{x}$. We may think of composition as using the functions P and $\sqrt{}$ sequentially, first applying P to x, then applying $\sqrt{}$ to the result.

Such use of arithmetical operators and the operation of composition to define new functions is so prevalent in algebra that we rarely give them second notice. For the record, however, we record the building of new functions in these ways in formal theorems:

Definition 1: Let f and g be functions and let x be in the domain of both functions. Then the functions $f + g$, $f - g$, fg and f/g are defined by the rules:

1. $(f + g)(x) = f(x) + g(x)$
2. $(f - g)(x) = f(x) - g(x)$
3. $(fg)(x) = f(x) \cdot g(x)$
4. $(f/g)(x) = f(x)/g(x)$, when $g(x) \neq 0$

Example 4: As an illustration of Definition 1, let $f(x) = \sqrt{x}$ and $g(x) = x^3$. Then $(f+g)(x) = \sqrt{x}+x^3$ for all x such that $x \geq 0$. That is, $(f + g)(x)$ is defined for any nonnegative value of x to be the sum of the square root of x and the cube of x. Hence, for example, $(f+g)(4) = 2 + 64 = 68$. As in the graph below, $f+g$ is a new function which we define pointwise to be the sum of the values of f and g.



Definition 2: Let f and g be functions, let x be in the domain of f, and $g(x)$ in the domain of f. Then the *composite function* $f \circ g$ is defined by the rule

$$(f \circ g)(x) = f(g(x)).$$

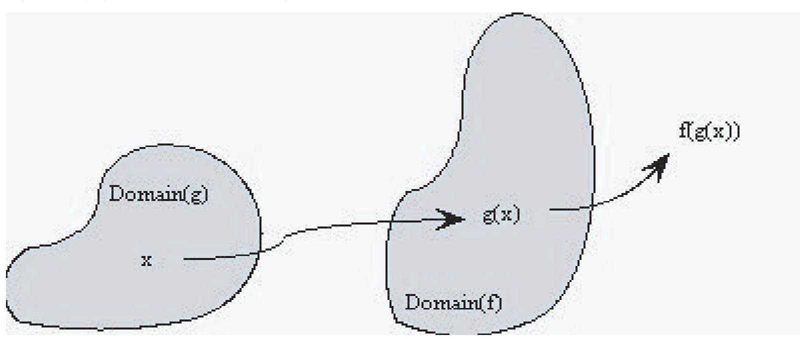
Using the language of mappings,

$$x \overset{g}{\longmapsto} g(x) \;\text{ and }\; g(x) \overset{f}{\longmapsto} f(g(x)).$$

We may think of the operation of *composition* as "joining the arrows", yielding

$$x \overset{f \circ g}{\longmapsto} f(g(x)).$$

The domain of $f \circ g$ is $\{x : g(x) \in \text{domain of } f\}$.



Example 5: Let $f(x) = x^2 - 4$ and $g(x) = x - 2$. Then

$$(f \circ g)(x) = f(g(x)) = g(x)^2 - 4 = (x-2)^2 - 4 = x^2 - 4x.$$

Example 6: Let $f(x) = 2 + \sqrt{x}$ and $g(x) = 4 - x^2$. Then

$$(f \circ g)(x) = 2 + \sqrt{g(x)} = 2 + \sqrt{4 - x^2}.$$

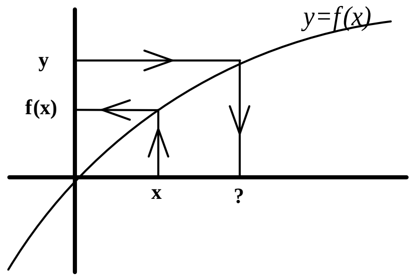
What is the domain of $f \circ g$? We must determine for which real numbers x is $g(x)$ in the domain of f. This requires that $g(x) \geq 0$, i.e. that $4 - x^2 \geq 0$. Thus $x^2 \leq 4$ or, finally, the domain of $f \circ g$ is $-2 \leq x \leq 2$.

Applet: Arithmetical Operations on Functions **Try it!**

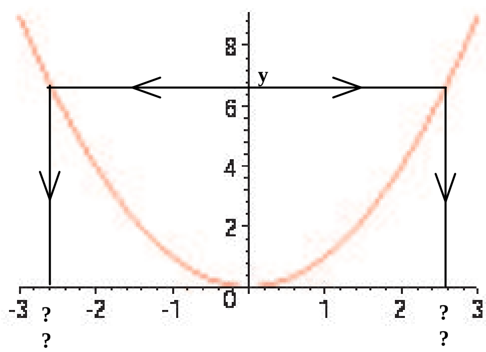
Inverse Functions: The final operation for building new functions from old is that of *taking inverses*. A familiar example is provided by the square root function $f(x) = \sqrt{x}$ and the square function $g(x) = x^2$. Their inverse relationship is exhibited (for $x > 0$) by the identities $\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$. In the form of equations we say that $y = x^2$ can be "solved for x" as $x = \sqrt{y}$; or, conversely, $y = \sqrt{x}$ can be "solved for x" as $x = y^2$. The relationship can also be viewed as reversing the arrow in the mapping that defines either of these two functions: the mapping $x \longmapsto y = x^2$ has the *inverse* mapping $y \longmapsto x = \sqrt{y}$.

Alternatively, if one considers a function to be a set of ordered pairs (x, y), then the corresponding *inverse relation* is the set of ordered pairs (y, x). If this inverse relation is also a function, then we call it the *inverse function.*

Or, if one thinks of the graph of a function f, computing values of $f(x)$ amounts to starting at a point x on the x-axis, tracing a vertical line to the graph and then a horizontal line to the y-axis. Computing the inverse relation of f amounts to reversing the direction of the arrows (see below), starting with a value y on the y-axis, tracing a horizontal line to the graph and then a vertical line to the x-axis. We will denote the "landing point" on the x-axis as $f^{-1}(y)$ and if f^{-1} defined by this rule is a function, we will refer to it as the *inverse of* f.



We hasten to dispose of the slight problem that the inverse relation of f is not always a function. In our picture there was a unique value on the x-axis corrsponding to a given value on the y-axis. But this need not be the case in general. We need look no further than the square function with which we began the discussion of inverse functions. Starting with a given value of y on the y-axis, which of the reverse arrows is to be followed to define an inverse function (see the following plot)? The problem arises, of course, from the fact that there are multiple values of x on the x-axis that are mapped to the *same* value of y. Under such circumstances the inverse mapping (or relation) is not a well-defined function. And we say that the function f does not have an inverse.



The solution is to restrict the domain of f so that the ambiguity does not arise. For the square function $f(x) = x^2$ we can do this, for example, by restricting the domain to the interval $0 \leq x \leq \infty$. Then there is a one-to-one correspondence between points x on the *positive* x-axis and points $y = f(x)$ on the *positive* y-axis. In such a case we say that the function f is 1–1, and then the inverse mapping as we have described it is unique.

Definition 3: A function f is said to be 1–1 if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. In other words different values of x are mapped to different values of y. Such a function is also said to pass the *horizontal line test*, in the sense that every line parallel to the x-axis intersects the graph of f in at most one point.

Theorem 4: If f is a 1–1 function then it has an inverse function which we will denote by f^{-1}. (Caution: do not confuse this with $1/f$, the reciprocal of f.) The domain of f^{-1} is the *range* of f; and the range of f^{-1} is the *domain* of f. The functions f and f^{-1} satisfy

$$y = f^{-1}(x) \qquad \text{if and only if} \qquad f(y) = x.$$

Example 7: We used the function $f(x) = x^2$ above as an example of a function that is not 1–1 on its full domain but that *is* 1–1 when its domain is restricted suitably. On the positive x-axis it is 1–1, and its inverse is called (the positive) square root of x. (We could have chosen the negative real axis as the restricted domain of f, and then the inverse function would have been the *negative* square root.

Example 8: Show that the function $f(x) = 5x + 2$ is 1–1 and find f^{-1}. Since the graph of f is a straight line that is not horizontal, it clearly passes the *horizontal line test*; thus f is 1–1 on its entire domain $(-\infty, \infty)$. Algebraically, we see that

$$f(x_1) = f(x_2) \Rightarrow 5x_1 + 2 = 5x_2 + 2 \Rightarrow x_1 = x_2.$$

To find the inverse function we follow the steps:

Step 1 $y = 5x + 2$

Step 2 Solve for x: $x = \frac{y-2}{5}$

Step 3 Reverse the roles of x and y: $y = \frac{x-2}{5}$

Step 4 Thus: $f^{-1}(x) = \frac{x-2}{5}$

Example 9: Find the inverse function of $f(x) = \sqrt{3x-1}$. To see that f is 1–1 we notice that

$$\sqrt{3x_1 - 1} = \sqrt{3x_2 - 1} \Rightarrow 3x_1 - 1 = 3x_2 - 1 \Rightarrow x_1 = x_2.$$

Then, following the same steps as in Example 8 we have

Step 1 $y = \sqrt{3x-1}$

Step 2 Solve for x: $y^2 = 3x - 1$, so $x = \frac{y^2+1}{3}$

Step 3 Reverse the roles of x and y: $y = \frac{x^2+1}{3}$

Step 4 Thus: $f^{-1}(x) = \frac{x^2+1}{3}$

Applet: Inverse Functions **Try it!**

Summary: This completes our survey of building blocks for defining new functions from old. Through arithmetic operations, composition, and taking inverses, we will build up our stable of functions—the so-called Elementary Functions—that suffice for much of calculus. We need to add only the trigonometric functions and the exponential and logarithm functions as “simple starting functions” for our building exercise. In the next section we handle the trigonometric functions. And the exponential and logarithm functions will follow thereafter.

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**