Functions and Their Graphs

At the heart of calculus lie two fundamental concepts—*function* and *limit*. From them are derived several additional basic concepts—*continuity*, *derivative*, and *integral*. It is the study of these several concepts and the mathematical techniques that accompany them that constitutes the subject of calculus. The applications of the concepts give calculus its power to model and understand dynamic systems. Understanding the concept of function thus becomes a first priority when studying calculus.

A function f is a correspondence between objects x in its domain and objects $f(x)$ in its range. For example the square function might be represented as $f(x) = x^2$, emphasizing the algebraic definition of the function. Or it might be represented as $f : x \to x^2$, emphasizing the correspondence or mapping between a number x and its square. The latter notation is very common in mathematics. In either case values of the function for given values of x are denoted by $f(x)$, as in $f(2) = 4$, $f(5) = 25$, or $f(-2.5) = 6.25$. In general, then:

Definition1: A function f on a set D into a set S is a rule that assigns a unique element $f(x)$ in S to each element x of D. The set D is called the *domain* of the function f and the subset $\{f(x) \in S : x \in D\}$ of S is called the *range* of f. It is common in this context to call x the *independent variable* because we assign its value, and y the *dependent variable* because we compute its value.

For instance, $f(x) = mx + b$, where m and b are real numbers, is a function. It takes any real number x and produces another real number $mx + b$. (If $m = 2$ and $b = 1$, then $f(1) = 3$.) Thus, the domain is the set of all real numbers; so is the range. The function f is a so-called *linear* function: the points $(x, f(x))$ lie on the line $y = mx + b$ in the plane.

Example 1: Let f be the function that maps each real number into its square. In this case the rule defining f may be given by an algebraic formula $f(x) = x^2$. The domain of f is the set of all real numbers. The range of f is the set of all non-negative real numbers.

Example 2: $g(x) = \sqrt{x}$, the function that maps real numbers into their real square roots. We take the domain of f to be the set of all non-negative real numbers. And then the range of f is also the set of all non-negative real numbers. (Why?)

Example 3: $f(x) = \sqrt{1 - x^2}$. A function is not completely defined, of course, until its domain is specified. In calculus it is conventional to assume that the domain is the set of all real numbers for which the value of the function is defined and is a real number. This would require that $0 \le 1 - x^2$. Thus the domain of f is

$$\{x : 0 \le 1 - x^2\} = \{x : x^2 \le 1\} = \{x : -1 \le x \le 1\}$$

Example 4: Consider the function $y = 1/x$. This time we have specified the function, one that maps real numbers into their reciprocals, by writing an equation relating the independent variable x and the dependent variable y. We could also have written $x \to 1/x$ to indicate the mapping, or we could have explicitly named the function $h(x) = 1/x$. Suffice it to say that we have many ways to specify the rule that defines the function, and we will use the one that is most convenient in a given situation. To complete this example we note that $1/x$ is not defined when $x = 0$. Since we have not otherwise specified the domain, we follow the *calculus domain convention* (cf Example 3) and take the domain to be the set of all real numbers different from zero, i.e. the set $\{x : x \ne 0\}$. This set would also commonly be written as the union of two intervals: $(-\infty, 0) \cup (0, \infty)$. (Remember that parentheses in the interval notation indicate that the end point of the interval is *not* included in the set, and square brackets mean that the end point *is* included.)

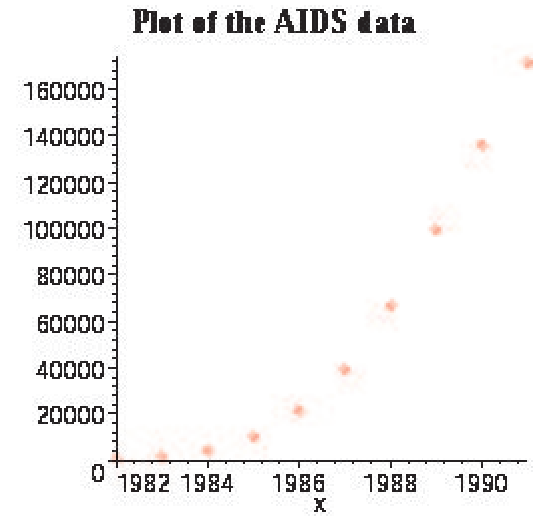
Functions can be represented algebraically, numerically, or graphically. When, for example, we state Newton's law of gravitation in the form

$$gravitationalforce = \frac{k}{r^2}$$

where k is a constant and r is the distance separating two bodies, we are giving an algebraic definition of a function. In this case the function expresses the inverse square law that relates the gravitational force to the distance r separating the two bodies. Often no such algebraic formula exists to express the functional relationship between two quantities but instead a table of numerical data is given. For example the spread of the AIDS virus in the United States during the first ten years following its discovery in 1981 is given in a table published by the Center for Disease Control:

Year	No of AIDS cases
1982	295
1983	1374
1984	4293
1985	10211
1986	21278
1987	39353
1988	66290
1989	98910
1990	135614
1991	170851

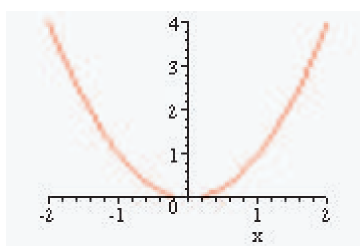
Such examples are common in science, where data is gathered in the laboratory or in the field. We can interpret this table as defining a function $AIDS(n)$ with finite domain, that gives the correspondence between the year n and the cumulative number of AIDS cases up to that year. Although the table is an efficient presentation of the AIDS function, it does not permit us to visualize its overall behavior. For that a graph of the data is most instructive.



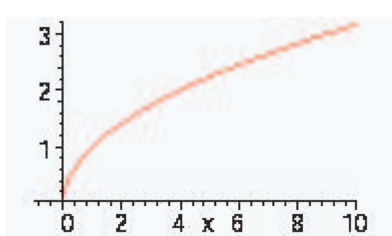
Alternatively, we can imagine a continuous function $f(x)$ such that $f(n) \approx AIDS(n)$ for each n and the graph of f runs smoothly between the data points. Such a function $f(x)$ is said to *model* the data. Finding a function that "fits" the data and that is also based on known scientific principles (in this example epidemiological principles) is an important problem in science. This is one of the most compelling reasons for studying the family of elementary functions of calculus—polynomials, rational functions, trigonometric functions, and exponential and logarithmic functions. Each of these classes of functions possesses its own unique and distinguishing properties, and choosing a function that models a given set of data involves matching those properties against the requirements of a particular scientific problem.

Graphing Functions: The graph of a function f often reveals its behavior more clearly than tabular or algebraic representations, thus familiarity with the graphs of selected basic functions is an important precursor to studying calculus. The graph of f is just the set of points $\{(x, y) : x \in \text{domain}, y = f(x)\}$. It is usually the geometrical picture of this set of points, plotted on an xy-coordinate system, that we have in mind when we use the term *graph*. For example the following are graphs with which you should be familiar:

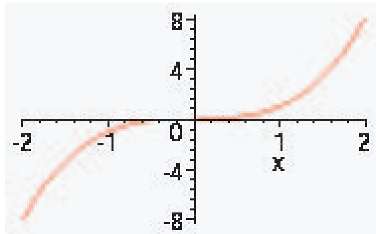
Example 5: $y = x^2$. The graph is the set of points $\{(x, x^2) : x \text{ is real}\}$. In drawing the picture we choose to plot the points in the range $-2 \leq x \leq 2$. We accept the fact that our geometrical picture of the graph does not cover the entire domain of the function. Normally we choose an x-range and y-range that reveal the most important behavior of the function.



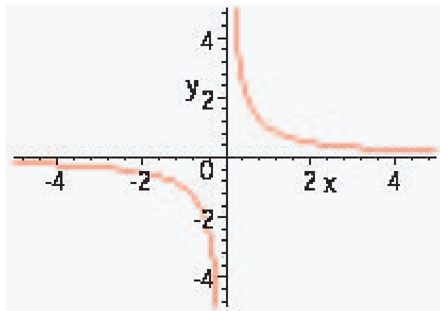
Example 6: $f(x) = \sqrt{x}$. In this example the domain of f is the interval $[0, \infty)$, and we choose the x-range of the plot to be $-1 \leq x \leq 10$.



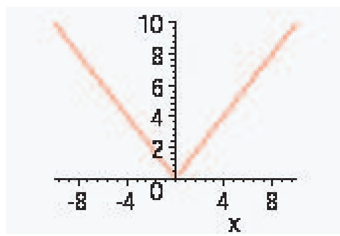
Example 7: $f(x) = x^3$. We choose the x-range $-2 \le x \le 2$.



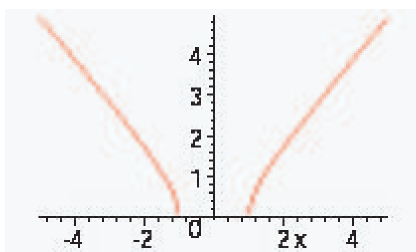
Example 8: Graph of $f(x) = \frac{1}{x}$.



Example 9: Graph of $f(x) = |x|$.



Example 10: Graph of $f(x) = \sqrt{x^2 - 1}$. Here we must have $0 \le x^2 - 1$. Hence the domain of f is $(-\infty, -1] \cup [1, \infty)$.

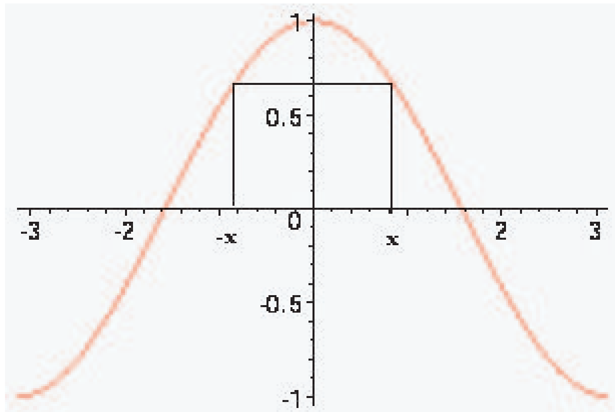


Graphs of functions may be drawn, albeit very tediously, by plotting many points and connecting them with a "smooth curve". Calculators or computers can be valuable tools in such pursuits. But the downside of drawing graphs in this way is that it obscures the most important properties of the functions and misdirects our attention to trivial computational details. A sketch of a graph, embodying the functions signature features, is often all that we need. Is the function positive or negative? Where does its graph cross the coordinate axes? Does it "shoot off to infinity"? Are there any gaps in its domain?

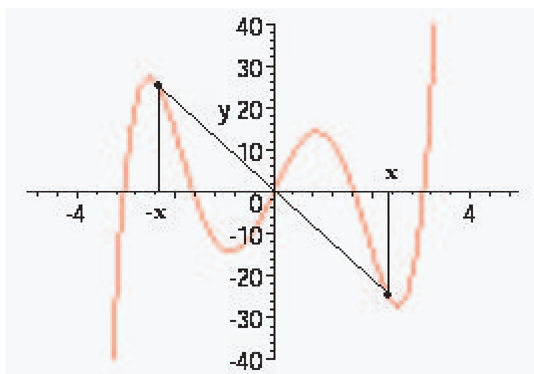
Fortunately, we will see that a little knowledge of graphs goes a long way. From the familiar graphs of a small number of basic functions we can recognize and sketch the graphs of many more functions that are related to them. We end this section with such an example—recognizing and exploiting symmetries in the graphs of functions.

Applet: Function Grapher **Try it!**

Even and Odd Functions; Symmetry and Reflections: In examples 5, 9 and 10, above, the graphs were symmetric about the y-axis. A point (x, y) lies on one of those graphs if and only if its mirror image $(-x, y)$ in the y-axis is also on the graph.



Examples 7 and 8 display a different kind of symmetry. A point (x, y) lies on the graph in Example 3 if and only if the point $(-x, -y)$ is also on the graph. The graph in this case is said to be "symmetric" about the origin.

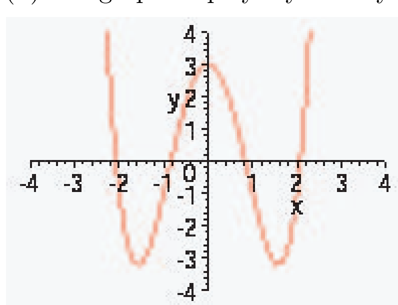


Definition 2: A function f is said to be an *even function* if $-x$ is in its domain whenever x is, and $f(-x) = f(x)$. Such a function is symmetric about the y-axis.

Definition 3: A function f is said to be an *odd function* if $-x$ is in its domain whenever x is, and $f(-x) = -f(x)$. Such a function is symmetric about the origin.

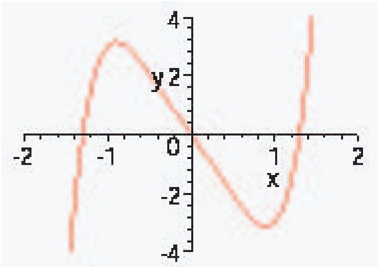
When a function is recognized as either an even or an odd function, its graph can be drawn exploiting the symmetry. Only half of the graph, for positive values of x, need be drawn. It can then be completed by reflecting in the y-axis (for an even function) or the origin (for an odd function).

Example 11: The function $f(x) = x^4 - 5x^2 + 3$ is an even function. Notice that replacing x by $-x$ does not change the function; i.e. $f(-x) = f(x)$. Its graph displays symmetry about the y-axis.



Example 12: On the other hand, the function $f(x) = 2x^5 - x^3 - 4x$ is an odd function. In this case it

is clear that $f(-x) = -f(x)$. Only odd powers of x appear in the expression, thus if x is replaced by $-x$ the minus sign can be "factored out".



Notice that an odd function, if it is defined for $x = 0$, must have the value zero there. For then it would have to satisfy $f(-0) = -f(0)$, and this implies that $f(0) = 0$. Why?

Applet: Symmetry, Odd and Even Functions **Try it!**

Exercises: Problems **Check what you have learned!**

Videos: Tutorial Solutions **See problems worked out!**