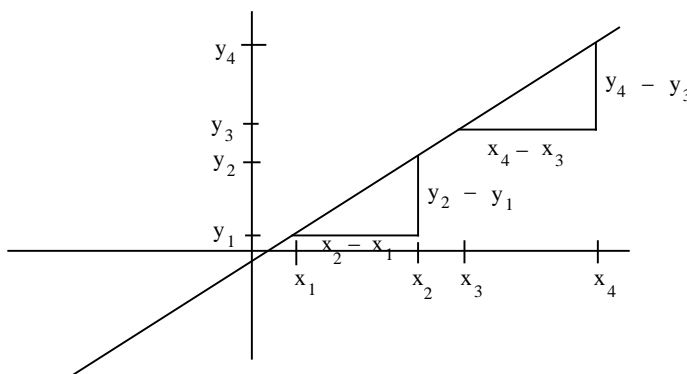


Lines in the Plane

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We have just seen in Section 1.1 that when we want to find a mathematical expression to represent a set of points in the plane, it is natural to begin by considering if the points fall on a line. But what are the various mathematical expressions for a line and how do we recognize a line from its equation?

Let's start by considering first a line in the plane. We want to develop an equation in x and y that will represent it. To do this, we label four points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , and (x_4, y_4) on the line.



Observe that the above triangles are similar because the corresponding angles are the same. Suppose now that given a right triangle whose hypotenuse has endpoints (x_1, y_1) and (x_2, y_2) we call $y_2 - y_1$ the *rise* and $x_2 - x_1$ the *run* where we form the differences in the same order (right point minus left, or vice versa, but in the same order for both the x and y coordinates). Then using this language, the similarity of the above triangles implies that the ratio of the rise to the run is the same for both these triangles:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

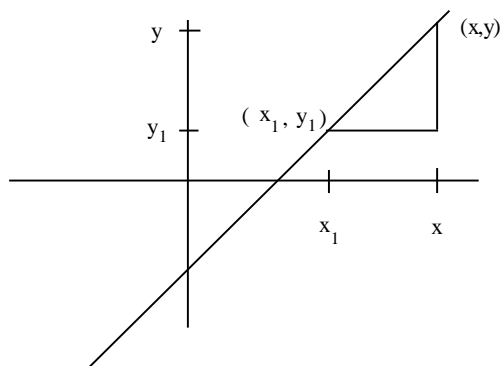
Thus, for any two different points on the line, the ratio of the rise to the run is the same constant m . Because this constant thereby characterizes the line, we give it a special name.

Definition 1: If a line passes through points (x_1, y_1) and (x_2, y_2) , where $x_1 \neq x_2$, we call $m = \frac{y_2 - y_1}{x_2 - x_1}$ the *slope* of the line.

Note that slopes are not defined for vertical lines (which have equations $x = k$ where k is a constant), and that the slope of a horizontal line is equal to 0. Also, reversing the order of the points (left minus right, instead of right minus left) does not change the value of the slope:

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}$$

Suppose now that we label two points on a line, (x, y) and (x_1, y_1) , where we assume that the point (x_1, y_1) has known coordinates x_1 and y_1 .



Then beginning with $\frac{y-y_1}{x-x_1} = m$, we can rewrite this equation to obtain an important form of equation of a line.

Point-Slope Form of Equation of a Line: Suppose that a line of slope m passes through the point (x_1, y_1) . Then the line has equation $y - y_1 = m(x - x_1)$.

If, instead of assuming we know the value of the slope and the coordinates of a point on the line, we assume that we know the coordinates of two points, we can still write the equation in point-slope form. First we calculate the slope, and then use it together with the coordinates of either one of the points to write an equation of the line in the point-slope form. We will illustrate this in the examples below.

If we start with the point-slope form of equation, solve for y and distribute m over the parentheses on the right-hand side, we thereby obtain the form $y = mx + b$. (Following the line of reasoning here, $b = y_1 - mx_1$, but this is not so important.) Setting $x = 0$, we see that this means that the line passes through the point $(0, b)$; the point b on the y -axis is called the y -intercept. Conversely, if a line of slope m passes through the point $(0, b)$, then from the point-slope form we see that the line has equation $y - b = m(x - 0)$, or $y = mx + b$.

Slope-Intercept Form of Equation of a Line: Suppose that a line of slope m has y -intercept b . Then the line has equation $y = mx + b$.

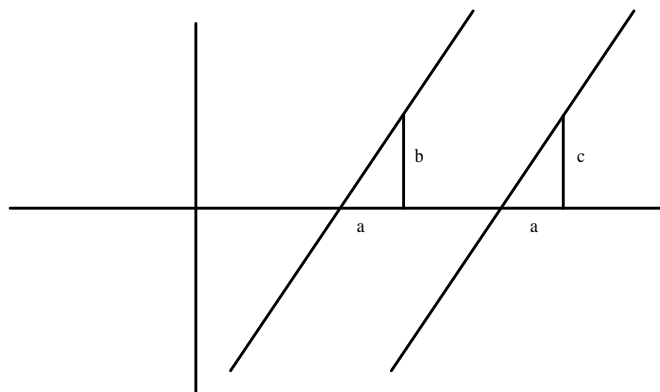
Finally, we see that by moving the terms involving x and y to one side of the equation and the constant terms to the other side, we can obtain an equation of the line in the so-called *general* form $ax + by = c$, where a, b, c are constants and the b here is not the same as the one above that we used to represent the y -intercept. Conversely, if we begin with an equation of the form $ax + by = c$, it is not too hard to see how to rewrite it to obtain, say, the slope-intercept form. (We will illustrate this in the examples below.) Hence, $ax + by = c$ is an equation of a line.

General Form of Equation of a Line: A line in the plane has an equation of the form $ax + by = c$ for some constants a, b, c ; and conversely, an equation of the form $ax + by = c$, where a, b , and c are constants, represents a line in the plane.

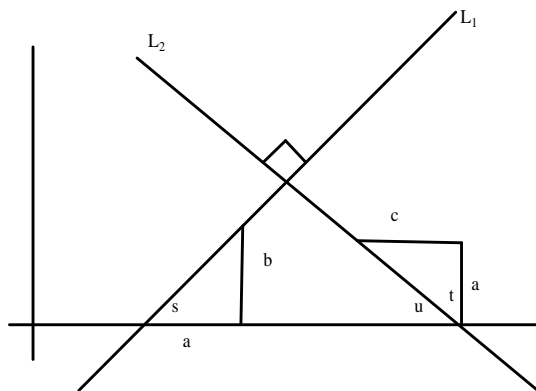
Example 1: To find an equation of the line through $(3, 4)$ and $(8, -6)$, we first find the slope of the line: $m = \frac{-6-4}{8-3} = -2$. Then from the point-slope form we obtain an equation $y - 4 = (-2)(x - 3)$.

Example 2: To find a general form of equation of the line through $(2, 7)$ and $(5, 6)$, we first find the slope: $m = \frac{6-7}{5-2} = -\frac{1}{3}$. Then we can arrive at a point-slope form: $y - 7 = -\frac{1}{3}(x - 2)$. And finally rewrite this equation in general form: $3y - 21 = 2 - x$, whence $x + 3y = 23$.

In closing our discussion of lines in the plane, we mention that two lines of slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$; moreover, the lines are perpendicular if and only if $m_1 = -1/m_2$. These relationships can be discerned from the sketches that follow.



For example, if the two lines above are parallel, then the two triangles are congruent. Hence, $b = c$. Thus, the slopes of the two lines are the same, namely, b/a .



Now, suppose the two lines L_1 and L_2 above are perpendicular. Then, in degrees, $u + t = 90$ and $u + s = 90$ implies $s = t$. Thus, the two little triangles are congruent and $b = c$. Hence, the slope of one line is $m_1 = b/a$ and the slope of the other is $m_2 = -a/b$. That is, $m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$.

Example 3: Find an equation of the line through $(1, 2)$ and parallel to the line with equation $5x - 7y = 2$. The slope of the latter line can be found by solving for y to obtain the slope-intercept form of the equation: $y = \frac{5}{7}x - \frac{2}{7}$. Thus, the line has slope $5/7$. So, in point-slope form, the equation of the line we seek is $y - 2 = (5/7)(x - 1)$.

Example 4: To find an equation of the line through $(1, 2)$ and perpendicular to the line with equation $5x - 7y = 2$, we note from the previous example that the slope must be $m = -7/5$ (the negative reciprocal of the slope of the line whose equation is given). Hence, the equation in point-slope form is $y - 2 = (-7/5)(x - 1)$.

Exercises: [Problems](#) **Check what you have learned!**

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