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Social positions in influence networks [1]

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Abstract

In this article we derive implications about social positions from a formal theory of social influence. The formal theory describes how, in a group of actors with heterogeneous initial opinions, a network of interpersonal influences enters into the formation of actors' settled opinions. We derive the following conclusions about a special form of structural equivalence. If actors are structurally equivalent in the network of interpersonal influences, then any dissimilarity of their initial opinions is reduced by the social influence process. If the social positions of actors are identical, i.e. if they have identical initial opinions and are structurally equivalent in the influence network, then they have identical opinions at equilibrium. If actors are not structurally equivalent in the network of interpersonal influences, then the social influence process does not necessarily reduce dissimilarities of initial opinions. We extend our analysis to consider automorphic equivalence.

Keywords: Networks; Positions; Influence; Consensus; Agreements

1. Introduction

In the network approach to social structure, the social positions of actors are revealed by the actors' patterns of relations with other actors, and a differentiated social structure is defined by the existence of actors who occupy different positions in networks of social relations. The most restrictive definition of social positions is based on structural equivalence which stipulates that two actors occupy the same social position if, and only if, they have identical relations with other actors (Lorrain and White, 1971). For

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example, in this restrictive definition two actors would be considered similar only if they had exactly the same set of friends.

Generalizations of structural equivalence have been vigorously pursued and have produced two lines of work. First, the qualitative definition of equivalence has been relaxed so that the *extent* of profile similarity might be assessed by taking into account one or several networks comprised of either binary or valued ties (Burt, 1976). Second, the restrictive focus on patterns of relations to and from particular actors has been relaxed to allow definitions of structurally equivalent network environments in which members are tied to the same *types* of actors; for example, along these lines, two actors in the authority structures of different organizations are similar if they are located in similar positions in the separate structures. The contributions of this latter line of work are numerous; see Borgatti and Everett (1992) and the literature cited therein. After nearly 20 years of continuous development, this research program has generated a large methodological literature on alternative approaches for describing social positions and a body of empirical evidence on associations between actors' positions and their opinions and behaviors.

The study of social positions is intended to elucidate not only the origins of individual opinion and behavior but also the structural foundations of *interpersonal agreement.* [2] Networks of endogenous interpersonal influences make the correspondence between social positions and interpersonal agreements somewhat more complex than might be thought. Endogenous interpersonal influences on opinions occur when actors take into account the opinions of other actors when forming their own opinions on an issue. If actors in otherwise similar social positions are subject to different interpersonal influences, then their opinions may diverge as a consequence of these influences. With a network of interpersonal influences at work, actors may have their opinions influenced (via intermediaries) by actors who are located elsewhere in the social structure. Hence, the settled opinions of actors are a potentially complex product of the entire *structure of social positions.*

In this article, we address the structural environment in which interpersonal agreements are formed; we do so with a formal theory of the opinion formation process that takes into account the network of endogenous interpersonal influences that may be shaping actors' opinions. We *begin* with social process and *deduce* the relevant positional concepts and relationships that bear on the reduction of variation in actors' opinions. This theoretical approach leads to conclusions about the structural prerequisites of interpersonal agreement, including consensus, and the interaction of social positions in producing such agreement.

The article has two main parts. First, we briefly review our theory of social influence. Second, we develop the implications of the theory for the relationship between social positions and interpersonal agreements.

[2] For example, Mizruchi (1993, p. 275) states: "Predicting the similarity of attitudes and behavior has been an important goal of network analysts since the 1950s. Until the mid-1970s, similarity of behavior was viewed almost exclusively as a function of social cohesion. In the 1970s, similarity began to be viewed as a function of structural equivalence. More recently, it has been viewed as a function of positional or role equivalence."

2. Social influence network theory

Typically, individuals form their opinions in a complex interpersonal environment in which influential opinions are in disagreement and subject to change. How opinions are formed and how consensus is reached under such complex circumstances is the subject of a formal theory that has been under development by social psychologists and mathematicians since the 1950s (French, 1956; Harary, 1959; DeGroot, 1974; Friedkin, 1986, Friedkin, 1990, Friedkin, 1991; Friedkin and Cook, 1990; Friedkin and Johnsen, 1990). The theory describes how a network of interpersonal influences enters into the process of opinion formation, and how this opinion formation process results in either a stable pattern of disagreement or group consensus.

Two equations describe the theory. One concerns the origins of actors' initial opinions,

$$\boldsymbol{y}^{(1)} = \boldsymbol{X}\boldsymbol{b} \tag{1}$$

and the other concerns the subsequent transformation of these initial opinions

$$\boldsymbol{y}^{(t)} = \alpha \boldsymbol{W}\boldsymbol{y}^{(t-1)} + (1-\alpha)\boldsymbol{y}^{(1)} \tag{2}$$

for $t = 2, 3, \ldots$, where $\boldsymbol{y}^{(t)}$ is an $n \times 1$ vector of individuals' opinions at time t, $\boldsymbol{X}$ is an $n \times k$ matrix of k exogenous variables, $\boldsymbol{b}$ is a $k \times 1$ vector of coefficients for the exogenous contributions, $\boldsymbol{W} = [w_{ij}]$ is an $n \times n$ matrix of endogenous interpersonal influences ($0 \leq w_{ij} \leq 1$, $\Sigma_j^n w_{ij} = 1$), and $0 \leq \alpha \leq 1$ is a scalar weight of the endogenous interpersonal influences. Within the framework of a simple recursive definition, this model stipulates that actors modify their opinions on an issue by forming a convex combination of influential opinions. Flows of interpersonal influence are established by the *repeated* responses of actors to the (possibly changing) influential opinions on the issue. The influence network for the group, $\boldsymbol{W}$, describes the pattern and magnitude of these direct endogenous interpersonal responses. However, actors not only are influenced endogenously by the opinions of other actors, but also are influenced exogenously, at each point in the process, by the conditions that have formed their *initial* opinions. The balance-of-forces (relative weight) of the endogenous and exogenous influences is described by α, the coefficient of social influence.

The balance-of-forces between exogenous and endogenous influences is a subtle but exceedingly important part of this model. The assumption of such a balance was introduced by Friedkin and Johnsen (1990) as a constraint that confines equilibrium opinions to the range of initial opinions. Without this constraint, the process may result in equilibrium opinions that 'breach' the range of initial opinions, a result that is grossly inconsistent with experimental evidence (Friedkin and Cook, 1990). Interestingly, the assumption of such a balance-of-forces also is consistent with the following speculation of Festinger:

> When a person or a group attempts to influence someone, does that person or group produce a totally new force acting on the person, one which had not been present prior to the attempted influence? Our answer is No – an attempted influence does not produce any new motivation or force. Rather, what an influence

attempt involves is the redirection of psychological forces which already exist. (Festinger, 1953, p. 237.)

Assuming equilibrium, under extreme conditions the model approaches the form of either (a) the classical linear model in which there are no endogenous interpersonal influences on opinions,

$$y^{(\infty)} = Xb \tag{3}$$

or (b) the consensus model of French, Harary, and DeGroot in which there are no ongoing exogenous influences on opinions,

$$y^{(\infty)} = W^{\infty} y^{(1)} \tag{4}$$

In the first case ($\alpha = 0$), endogenous interpersonal influences are absent and actors' equilibrium opinions correspond to their initial opinions. In the second case ($\alpha = 1$), endogenous interpersonal influences are dominant and exogenous conditions have no substantive impact on opinions other than in forming the actors' initial opinions on an issue. Thus, apart from these extreme circumstances, the opinions that are formed at each point in the process (Eq. (2)), reflect the *competing* influences of the personal circumstances of actors and the opinions (and, therefore, indirectly also the circumstances) of their significant others.

In general, assuming equilibrium, actors' settled opinions may be described thusly

$$y^{(\infty)} = V y^{(1)} \tag{5}$$

where

$$v_{ij} = \begin{cases} \tilde{v}_{ij} & \text{for } \alpha < 1 \\ w_{ij}^{(\infty)} & \text{for } \alpha = 1 \end{cases} \tag{6}$$

in which $\tilde{V} = [\tilde{v}_{ij}]$,

$$\tilde{V} = (I - \alpha W)^{-1}(1 - \alpha) \tag{7}$$

and

$$W^{\infty} = \left[w_{ij}^{(\infty)} \right] \tag{8}$$

In this reduced-form equation, the coefficients of $V = [v_{ij}]$ describe the *total interpersonal effect* of the initial opinion of actor j on the equilibrium opinion of actor i ($0 \le v_{ij} \le 1$, $\Sigma_j^n v_{ij} = 1$). [3]

We illustrate the model and several of our conclusions about social positions with an example shown in Fig. 1. The example concerns an influence network among eight actors who are divided into two cliques that have initially different orientations on an issue. The influences among the members are indicated by the coefficients on the directed lines connecting pairs of actors. For example, the line from actor 1 to actor 2 has the value 0.55, which indicates the relative influence of actor 1 on actor 2. Because

[3] Elsewhere we have shown that W^{∞} is the limiting $\tilde{V}$ for $\alpha \to 1$ (Friedkin and Johnsen, 1990).

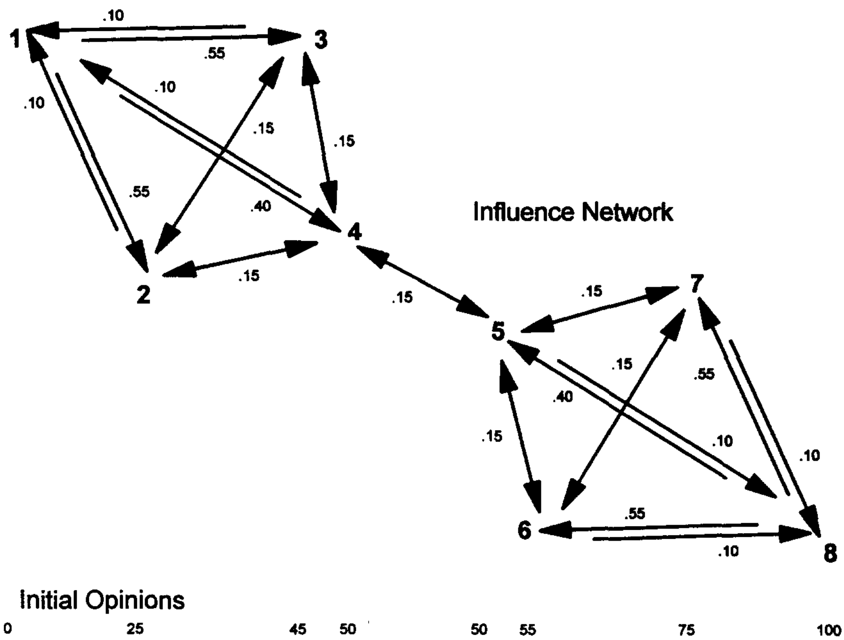

Fig. 1. Illustrative case.

interpersonal influence is treated as a finite distributed resource, the influences upon an actor sum to 1.0. Self-weights are allowed; although the loops on the points are not shown, the values of these self-weights are readily calculated. For example, the interpersonal influences upon actor 3 sum to 0.85 and, therefore, the value of the loop for actor 3 is 0.15. The matrix of interpersonal influences for the group is:

$$W=\begin{bmatrix} 0.70 & 0.10 & 0.10 & 0.10 & 0 & 0 & 0 & 0 \\ 0.55 & 0.15 & 0.15 & 0.15 & 0 & 0 & 0 & 0 \\ 0.55 & 0.15 & 0.15 & 0.15 & 0 & 0 & 0 & 0 \\ 0.40 & 0.15 & 0.15 & 0.15 & 0.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15 & 0.15 & 0.15 & 0.15 & 0.40 \\ 0 & 0 & 0 & 0 & 0.15 & 0.15 & 0.15 & 0.55 \\ 0 & 0 & 0 & 0 & 0.15 & 0.15 & 0.15 & 0.55 \\ 0 & 0 & 0 & 0 & 0.10 & 0.10 & 0.10 & 0.70 \end{bmatrix}$$

The two cliques of the network consist of actors {1, 2, 3, 4} and {5, 6, 7, 8} and they are joined by a bridge between actors 4 and 5. Each clique contains a leader (actors 1 and 8) plus three other actors who accord one another equal influence. The vector of initial opinions of the actors,

$$y^{(1)}=[0 \quad 25 \quad 45 \quad 50 \quad 50 \quad 55 \quad 75 \quad 100]^T$$

Table 1
Illustrative equilibrium opinions

Actor	$y^{(1)}$	$y^{(\infty)}$			
		$\alpha = 0.250$	$\alpha = 0.500$	$\alpha = 0.750$	$\alpha = 0.099$
1	0	3.2	6.9	12.2	48.8
2	25	23.1	21.2	20.0	48.9
3	45	38.1	31.2	25.0	48.9
4	50	43.9	37.8	32.5	49.2
5	50	56.1	67.2	67.5	50.8
6	55	61.9	68.8	75.0	51.1
7	75	76.9	78.8	80.0	51.1
8	10	96.8	93.1	87.8	51.2

Note. These results are based on the network shown in Fig. 1.

indicates that the two cliques tend to differ in their initial opinions (one clique has higher values than the other) and that the opinions of the two leaders are the most polarized.

The equilibrium opinions of the actors depend on the level of α, the coefficient of social influence, which determines the relative weight of the endogenous and exogenous influences upon opinions. Table 1 presents four sets of outcomes, where each set corresponds to a different weight of endogenous influence. We will draw on these results later on.

3. Social positions

The theory suggests that there are two basic types of social position. Social positions occur in $\boldsymbol{W}$, the network of endogenous interpersonal influences, and in $\boldsymbol{X}$, the matrix of exogenous variables that affect actors' opinions. These endogenous and exogenous social positions have distinct roles in the development of individual and collective outcomes. In this section, we describe these two types of social position and analyze their bearing on interpersonal agreements.

3.1. Social positions in X

Exogenous social positions are defined by a subset of variables in $\boldsymbol{X}$ (Eq. (1)), i.e. the matrix that contains all exogenous determinants of the opinions of actors. [4] These social positions include individual attributes such as gender, age, and socioeconomic status (Blau, 1977). They also may correspond to ubiquitous roles (physician, father, husband), to local statuses (gang member, community leader), or to locations in other networks of social relations (friendship networks, authority structures).

A broader viewpoint on exogenous social positions is based on the profile similarity of actors *across* the variables in $\boldsymbol{X}$. In this broader view, actors occupy more or less

[4] As a theoretical construct, $\boldsymbol{X}$ contains all the conditions that affect the initial opinions of actors. A column in $\boldsymbol{X}$ may contain a random component. A statistical model may be derived by partitioning $\boldsymbol{X}$ (along with the corresponding vector of effects $\boldsymbol{b}$) into an observed part and an unobserved (error) part.

similar locations in a multidimensional space of exogenous variables that affect their opinions. McPherson and Ranger-Moore (1991) refer to these locations as positions in Blau-Space. We define two actors (ij) as *exogenously equivalent* if they have identical profiles of values across the K variables in $\boldsymbol{X}$. [5]

Close proximity in Blau-Space indicates an approximate equivalence of those exogenous conditions that affect the opinions of actors. Actors who occupy the same locations in Blau-Space have identical initial opinions. Whether or not such actors also have identical equilibrium opinions depends on the network of endogenous influences. This influence network potentially disrupts the correspondence between the personal circumstances of actors $(\boldsymbol{X})$ and their settled opinions $(\boldsymbol{y}^{(\infty)})$; and, therefore, actors with identical initial opinions need not have identical settled opinions. The notion that actors in similar circumstances have similar opinions follows from the classical assumption of actor independence. When this assumption of independence is relaxed to allow for actors who are responding not only to their own circumstances, but also to the responses of other actors, then this 'common fate' hypothesis falls to the ground.

For example, consider the bridging actors {4, 5} in our illustrative network. These two actors have *identical* initial opinions, and the effect of the network of endogenous influences is to *polarize* their opinions (see Table 1). The distance between the equilibrium opinions of the two actors {4, 5} is 12.2, 24.4, and 35.0 for α-values of 0.25, 0.50, and 0.75, respectively. Only under the full potential weight of the endogenous network (i.e. for $\alpha \to 1$) is the polarization of opinions reduced. In short, although the exogenous equivalence of two actors implies initial $(\boldsymbol{y}^{(1)})$ agreement, such shared location and agreement does not necessarily imply equilibrium $(\boldsymbol{y}^{(\infty)})$ agreement.

3.2. Social positions in W and V

Social positions also occur in networks of endogenous interpersonal influence. These positions are based on *endogenous equivalencies* of direct or total interpersonal effects, as opposed to the *exogenous equivalencies* of actors' profiles in $\boldsymbol{X}$. In this section we define several types of endogenous social positions that have a bearing on the account of interpersonal agreements.

(1) Two actors i and j are defined as *structurally equivalent* in $\boldsymbol{W}$ if $w_{ik} = w_{jk}$ for all $k \neq i,j$. Actors with such vectors are structurally equivalent in that they are subject to an identical set of direct interpersonal effects from all other members of the group, excluding themselves. [6] A measure of the degree of two actors structural equivalence in $\boldsymbol{W}$ is

[5] Social positions that are defined on the basis of concrete social networks (apart from the network of endogenous social influences) may be viewed in one of two ways. The resulting positions may be treated as variables within $\boldsymbol{X}$ (i.e. dummy variables) or they may serve as an empirical proxy of the profile similarity of actors in $\boldsymbol{X}$.

[6] Other definitions of structural equivalence, which include the influences of actors i and j upon others, are not theoretically warranted by our approach. Although actors may vary in the extent of their similarity of influences *upon* other actors, such similarity does not appear to have implications for interpersonal agreements.

$$S^{(w)} = \left[s_{ij}^{(w)} \right] \tag{9}$$

where

$$s_{ij}^{(w)} = 1 - \left(\frac{\Sigma_k (w_{ik} - w_{jk})^2}{\Sigma_k w_{ik}^2 + \Sigma_k w_{jk}^2} \right)^{1/2} \tag{10}$$

for $k \neq i,j$ and $\Sigma_k w_{ik}^2 + \Sigma_k w_{jk}^2 \neq 0$; otherwise $s_{ij}^{(w)} = 1$. This measure, which varies between one and zero, will equal one when actors i and j are structurally equivalent and will equal zero when the influences upon the two actors are disjoint.

We show (see Appendix A) that if two actors are structurally equivalent in $\boldsymbol{W}$, then the difference between the opinions of the two actors at time $t+1$ is

$$\left(y_i^{(t+1)} - y_j^{(t+1)} \right) = \left[(1-\alpha)\left(1 + \alpha\gamma_w + \alpha^2\gamma_w^2 + \cdots + \alpha^{t-1}\gamma_w^{t-1}\right) + \alpha^t\gamma_w^t \right] \times \left(y_i^{(1)} - y_j^{(1)} \right) \tag{11}$$

for $t = 1, 2, \ldots$, where $\gamma_w = w_{ii} - w_{ji} = w_{jj} - w_{ij}$. When $\alpha\gamma_w = 1$, there is no change of the initial opinions of actors i and j. When $\alpha\gamma_w = 1$, there are no equilibrium opinions because actors i and j continually interchange their initial opinions. Of particular interest is the case of $|\alpha\gamma_w| < 1$, where we show that the influence process reduces an initial difference of opinion between the two actors by an amount that depends on $(1 - \alpha/1 - \alpha\gamma_w)$.

$$\left(y_i^{(t+1)} - y_j^{(t+1)} \right) = \left[\left(\frac{1-\alpha}{1-\alpha\gamma_w} \right) + \left(1 - \frac{1-\alpha}{1-\alpha\gamma_w} \right) (\alpha\gamma_w)^t \right] \left(y_i^{(1)} - y_j^{(1)} \right) \tag{12}$$

.Here, for $t \to \infty$, we arrive at the equilibrium difference of opinion

$$\left(y_i^{(\infty)} - y_j^{(\infty)} \right) = \left(\frac{1-\alpha}{1-\alpha\gamma_w} \right) \left(y_i^{(1)} - y_j^{(1)} \right) \tag{13}$$

For instance, in Fig. 1 actors {2, 3} and {6, 7} are subsets of structurally equivalent actors in $\boldsymbol{W}$. In both of these subsets the initial opinion difference is 20, and $\gamma_w = 0$. This initial difference of opinion is reduced to 15 for $\alpha = 0.25$, to 10 for $\alpha = 0.50$, and to 5 for $\alpha = 0.75$.

(2) Two actors are defined as *structurally equivalent* in $\boldsymbol{V}$ if $v_{ik} = v_{jk}$ for all $k \neq i,j$, where $\boldsymbol{V}$ is the matrix of total interpersonal effects that transform initial into equilibrium opinions (Eq. (5)). Actors with such vectors in $\boldsymbol{V}$ are subject to an identical set of total interpersonal effects from all other members of the groups, excluding themselves (see also Footnote 6). A measure of the degree of two actors structural equivalence in $\boldsymbol{V}$ is

$$S^{(v)} = \left[s_{ij}^{(v)} \right] \tag{14}$$

where

$$s_{ij}^{(v)} = 1 - \left(\frac{\Sigma_k (v_{ik} - v_{jk})^2}{\Sigma_k v_{ik}^2 + \Sigma_k v_{jk}^2} \right)^{1/2} \tag{15}$$

for $k \neq i,j$ and $\sum_k \nu_{ik}^2 + \sum_k \nu_{jk}^2 \neq 0$; otherwise $s_{ij}^{(\nu)} = 1$. This measures varies between zero and one. It will equal one when actors i and j are structurally equivalent in V and will equal zero when the influences upon the two actors are disjoint.

If two actors are structurally equivalent in W, then the two actors also are structurally equivalent in V.[7] If two actors are structurally equivalent in V, then

$$\begin{aligned}\left(y_i^{(\infty)} - y_j^{(\infty)}\right) &= \nu_{ii} y_i^{(1)} + \nu_{ij} y_j^{(1)} - \nu_{ji} y_i^{(1)} - \nu_{jj} y_j^{(1)} \\ &= (\nu_{ii} - \nu_{ji}) y_i^{(1)} - (\nu_{jj} - \nu_{ij}) y_j^{(1)} \\ &= \gamma_\nu \left(y_i^{(1)} - y_j^{(1)} \right)\end{aligned}$$

for $|\gamma_\nu| \leq 1$, where $\gamma_\nu = \nu_{ii} - \nu_{ji} = \nu_{ji} = \nu_{jj} - \nu_{ij}$.[8] In the special case of $\gamma_\nu = 0$ (where $\alpha = 1$), the row vectors of actors i and j are identical, the two actors are subject to an identical set of total interpersonal effects, and they will have identical equilibrium opinions. We say that such actors i and j are *force equivalent*.

In our illustrative network, for $\alpha = 1$ the total effects matrix is

$$V = \begin{bmatrix} 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \\ 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \\ 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \\ 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \\ 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \\ 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \\ 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \\ 0.31 & 0.06 & 0.06 & 0.07 & 0.07 & 0.06 & 0.06 & 0.31 \end{bmatrix}$$

and the equilibrium opinions are $y^{(\infty)} = [50\ 50\ 50\ 50\ 50\ 50\ 50\ 50]^{\mathrm{T}}$. For $\alpha = 1$, all the actors occupy a single (force equivalent) position and, therefore, are expected to be in agreement. In general, if $\alpha = 1$ and the structure of W is regular or centered, then all the actors in such networks will be force equivalent and the outcome of the influence process will be global consensus regardless of the distribution of initial opinions.[9]

If W is *not* regular or centered, but consists of disconnected regular or centered subnets, and $\alpha = 1$, then a number of force equivalent positions will emerge. Each of these positions will correspond to a maximally complete regular or centered subnetwork

[7] If actors i and j are structurally equivalent in W (i.e. $w_{ik} = w_{fk}$ for all $k \neq i,j$), then so are they in every power of W. Hence, they will be structurally equivalent in V when $\alpha = 1$, because $V = W^\infty$, and when $\alpha < 1$, because $V = (I + \alpha W + \alpha^2 W^2 + \cdots + \alpha^k W^k + \cdots)(1 - \alpha)$.

[8] If actors i and j are structurally equivalent in V (i.e. $\nu_{ik} = \nu_{jk}$ for all $k \neq i,j$), then $\nu_{ii} + \nu_{ij} = \nu_{ji} + \nu_{jj}$.

[9] For the definitions of regular and centered influence, we draw on the theory of diagraphs (Harary et al., 1965) and Markov chains (Kemeny and Snell, 1960). A network is *disconnected* if its membership can be partitioned into two or more groups between which no influence relations exist; otherwise it is *connected*. A connected network is *strong* (ergodic) if every member has direct or indirect influence on all other members. A strong network is *regular* (aperiodic or acyclic) if some power of its matrix presents entirely positive entries. Otherwise, the network is *periodic*. A connected network is *unilateral* if, for all pairs of members, at least one member of a pair has direct or indirect influence on the other member. We refer to a unilateral network as *centered* if it contains a single regular subnetwork of size $n \geq 1$ whose members directly or indirectly influence all other network members.

and will consist of agreeing actors. In such a case, the matrix of total interpersonal effects may be rearranged in block diagonal form, for example,

$$V = \begin{bmatrix} c_1 & c_2 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & c_4 & c_5 & 0 \\ 0 & 0 & c_3 & c_4 & c_5 & 0 \\ 0 & 0 & c_3 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 \end{bmatrix}$$

where each block is composed either of a single actor or a larger subset of actors involved in a regular or centered subnetwork: if V has this form, then W must have the same pattern of zeroblocks. Thus, for $\alpha = 1$ the endogenous social structure of the group may be represented succinctly as a *blockmodel* that is defined by the pattern of zeroblocks in W or V. [10]

For $\alpha < 1$, particular actors may or may not be in agreement at equilibrium. [11] The expectation for $\alpha \ll 1$, is a pattern of disagreement. However, relatively homogeneous clusters of actors may arise as a consequence of the influence process. For example, in the illustrative results in Table 1, the range of opinions *within each subgroup* is reduced from 50 to either 40.7, 30.9, 20.3, or 0.4 depending on the relative weight of endogenous influences ($\alpha = 0.25$, 0.50, 0.75, and 0.999, respectively). This homogenization of opinions within the two subgroups is occurring even as the difference of the average opinions between the two subgroups is increasing, i.e. interpersonal influences are polarizing the two subgroups.

(3) Finally, two actors are said to be *automorphically equivalent* in an influence network M (i.e. V or W) if there is a permutation matrix P such that $PMP^T = M$ which maps one actor to the other (Borgatti and Everett, 1992). The matrix P represents an automorphism of M, which can be thought of either as a renaming of the actors or as a shifting of the actors into new positions. This permutation may alter the identities of the actors whom they influence, and who are influencing them, but does not alter the structure of the network. Subsets of actors who may be permuted into each other's positions without altering the structure of the network are called *orbits*, and the members of the same orbit are said to be automorphically equivalent. The orbits in our illustrative network are {1, 8}, {2, 3, 6, 7}, and {4, 5}.

We now show that for $0 < \alpha < 1$, an automorphism of W is an automorphism of V, and vice versa. For a permutation matrix Q we have

[10] There is an extensive literature on blockmodels of social structure. On the concept and use of such blockmodels see White et al. (1976) and Arabie et al. (1978).

[11] For $\alpha < 1$, $(I - \alpha W)$ is always invertible because $\alpha^{-1} > 1$ and hence cannot be an eigenvalue of W, which is stochastic. Global consensus can appear only if there was consensus at the start of the process. Because $y^{(\infty)} = Vy^{(1)}$ and each of the rows in V^{-1} sum to one, if the outcome is consensus then $V^{-1}y^{(\infty)} = y^{(\infty)} = y^{(1)}$.

$$QQ^T = I = Q^TQ \tag{17}$$

From the equation $V = (I - \alpha W)^{-1}(1 - \alpha)$ we readily derive the equivalent equation

$$QVQ^T(I - \alpha QWQ^T) = (1 - \alpha)I \tag{18}$$

(a) If Q represents an automorphism of W, i.e. $QWQ^T = W$, then from Eq. (18) we obtain

$$QVQ^T(I - \alpha W) = (1 - \alpha)I \tag{19}$$

or

$$QVQ^T = (I - \alpha W)^{-1}(1 - \alpha) = V \tag{20}$$

which says that Q represents an automorphism of V.

(b) If Q represents an automorphism of V, i.e. $QVQ^T = V$, then from Eq. (18) we obtain

$$V(I - \alpha QWQ^T) = (1 - \alpha)I \tag{21}$$

or

$$(I - \alpha W)^{-1}(1 - \alpha) = V = (I - \alpha QWQ^T)^{-1}(1 - \alpha) \tag{22}$$

which leads to

$$I - \alpha W = I - \alpha QWQ^T \tag{23}$$

which for $\alpha \neq 0$ gives

$$W = QWQ^T \tag{24}$$

which says that Q is an automorphism of W.

Moreover, if there is an automorphism P of V, then $Py^{(\infty)} = (PVP^T)Py^{(1)} = VPy^{(1)}$. Hence, if those actors who are permuted into new positions in V *carry their initial opinions with them*, then these actors' equilibrium opinions will be carried also. A corollary result is that if automorphically equivalent actors in W have identical initial opinions (i.e. if all members of an orbit have the same initial opinion), then the orbit also will have identical equilibrium opinions.

3.3. Summary

Shared position in Blau-Space (X) is not necessary or sufficient for interpersonal agreement at equilibrium. However, if actors who share the same position in Blau-Space also are structurally equivalent in the network of direct endogenous influences (W), then any initial agreement of these actors will be maintained by the process of opinion formation, and any disagreement between them will be reduced as the weight (α) of the endogenous influences in the process is increased. Along the same lines, if actors who share the same position in Blau-Space are automorphically equivalent in W, then these actors will have identical equilibrium opinions. In short, if shared locations in Blau-Space imply endogenous equivalence (either structural or automorphic), then exogenous equivalence directly translates into an interpersonal agreement or a reduction of the opinion discrepancy. In the absence of endogenous equivalence, exogenous equivalence does not reliably predict shared agreement. Actors in proximate social positions in Blau-Space may differ substantially in their equilibrium opinions.

4. Discussion

The present paper contributes to the mathematical foundations of a structural theory of social positions. It does so by approaching the subject from the standpoint of a network theory of social influence. Within the framework of this approach, interactionist and structural orientations towards positional effects are combined, at least to the extent that both structure and process enter into the account of the formation of opinions. Structure enters into the account in two forms: a set of exogenous conditions that influence actors' opinions and a network of interpersonal influences. We have shown that these two forms of structure importantly interact in the development of interpersonal agreements.

Our approach departs from a particular structuralist viewpoint, propounded by Merton (1968), that eschews a close attention to the interpersonal influence process. Merton argued that structural analysis should not be concerned with the manner in which actors attempt to integrate and reconcile conflicting influences upon them; instead, he argued that theoretical work should focus on the conditions that reduce the problems of role ambiguity and conflict for the occupants of social positions. In contrast to this viewpoint on structural analysis, we derive an analysis of social positions from a *social process* model of opinion formation, which specifies how actors tend to form opinions under conditions of conflicting and changing influential opinions.

Our approach does not assume a correspondence between actors' positions in Blau-Space and their positions in the network of endogenous interpersonal influences; however, such a correspondence is theoretically useful. Where social positions in Blau-Space powerfully affect opinions, actors in the same positions will have similar initial orientations on issues. But if the joint occupants of positions in Blau-Space vary in the pattern of influences upon them, then their equilibrium opinions may differ. Matters become more straightforward if the joint occupants of social positions in Blau-Space also have identical patterns of interpersonal influences; when this is the case, any initial agreements among them will be maintained and any disagreements among them will be reduced to some extent.

An approach that takes into account the social process of opinion formation is useful because it applies to situations where the joint occupants of a status (i.e. actors with identical values on a variable in $\boldsymbol{X}$) are also importantly differentiated (i.e. they have dissimilar values on other variables in $\boldsymbol{X}$), so that initial disagreements frequently arise among some of the actors. Actors in such a differentiated status are faced with a considerable problem of reconciling or mitigating their differences especially where there are pressures toward uniformity. Within the weak and often confusing constraints provided by role definitions, actors must work hard at negotiating shared understandings. The present theory describes a mechanism by which such agreements are formed.

Appendix A

In this appendix we first show inductively that if two actors i and j are structurally equivalent in $\boldsymbol{W}$, then

$$\left(y_i^{(t+1)}-y_j^{(t+1)}\right)=\left[(1-\alpha)\left(1+\alpha\gamma_w+\alpha^2\gamma_w^2+\cdots+\alpha^{t-1}\gamma_w^{t-1}\right)+\alpha^t\gamma_w^t\right]\times\left(y_i^{(1)}-y_j^{(1)}\right) \tag{A1}$$

where $\gamma_w=w_{ii}-w_{ji}=w_{jj}-w_{ij}$, $t=1, 2, \ldots$.

If actors i and j are structurally equivalent in $\boldsymbol{W}$ (i.e. $w_{ik}=w_{jk}$ for all $k\neq i,j$), then $w_{ii}+w_{ij}=w_{ji}+w_{jj}$ and so

$$w_{ii}-w_{ji}=w_{jj}-w_{ij}\equiv\gamma_w \tag{A2}$$

where $|\gamma_w|\leq 1$ because $\max|w_{ii}-w_{ji}|=\max|w_{jj}-w_{ij}|=1$.

If the initial opinions of actors $\boldsymbol{y}^{(1)}$ has $y_i^{(1)}-y_j^{(1)}\equiv\delta^{(1)}$ then in $\boldsymbol{y}^{(2)}=\alpha\boldsymbol{W}\boldsymbol{y}^{(1)}+(1-\alpha)\boldsymbol{y}^{(1)}$ we have

$$\delta^{(2)}\equiv y_i^{(2)}-y_j^{(2)}=[1-\alpha+\alpha\gamma_w]\delta^{(1)} \tag{A3}$$

in $\boldsymbol{y}^{(3)}=\alpha\boldsymbol{W}\boldsymbol{y}^{(2)}+(1-\alpha)\boldsymbol{y}^{(1)}$ we have

$$\delta^{(3)}\equiv y_i^{(3)}-y_j^{(3)}=\left[(1-\alpha)(1+\alpha\gamma_w)+\alpha^2\gamma_w^2\right]\delta^{(1)} \tag{A4}$$

in $\boldsymbol{y}^{(4)}=\alpha\boldsymbol{W}\boldsymbol{y}^{(3)}+(1-\alpha)\boldsymbol{y}^{(1)}$ we have

$$\delta^{(4)}\equiv y_i^{(4)}-y_j^{(4)}=\left[(1-\alpha)\left(1+\alpha\gamma_w+\alpha^2\gamma_w^2\right)+\alpha^3\gamma_w^3\right]\delta^{(1)} \tag{A5}$$

and in $\boldsymbol{y}^{(5)}=\alpha\boldsymbol{W}\boldsymbol{y}^{(4)}+(1-\alpha)\boldsymbol{y}^{(1)}$ we have

$$\delta^{(5)}\equiv y_i^{(5)}-y_j^{(5)}=\left[(1-\alpha)\left(1+\alpha\gamma_w+\alpha^2\gamma_w^2+\alpha^3\gamma_w^3\right)+\alpha^4\gamma_w^4\right]\delta^{(1)} \tag{A6}$$

Thus we have

$$\begin{aligned}\delta^{(t)}&\equiv y_i^{(t)}-y_j^{(t)}\\&=\left[(1-\alpha)\left(1+\alpha\gamma_w+\alpha^2\gamma_w^2+\cdots+\alpha^{t-2}\gamma_w^{t-2}\right)+\alpha^{t-1}\gamma_w^{t-1}\right]\delta^{(1)}\end{aligned} \tag{A7}$$

for $t=2, 3, 4, 5$.

Inductively, assume that Eq. (A7) holds for a value of $t\geq 2$. Then in $\boldsymbol{y}^{(t+1)}=\alpha\boldsymbol{W}\boldsymbol{y}^{(t)}+(1-\alpha)\boldsymbol{y}^{(1)}$ we have

$$\begin{aligned}\delta^{(t+1)}&\equiv y_i^{(t+1)}-y_j^{(t+1)}\\&=\left[(1-\alpha)\left(1+\alpha\gamma_w+\alpha^2\gamma_w^2+\cdots+\alpha^{t-1}\gamma_w^{t-1}\right)+\alpha^t\gamma_w^t\right]\delta^{(1)}\end{aligned} \tag{A8}$$

which proves Eq. (A1). Note that if the initial opinions of actors i and j are identical, i.e. $\delta^{(1)}=0$, then their opinions remain identical at every subsequent time period.

Case 1. If $|\alpha\gamma_w|<1$ then

$$1+\alpha\gamma_w+\alpha^2\gamma_w^2+\cdots+\alpha^{t-1}\gamma_w^{t-1}=(1-\alpha\gamma_w)^{-1}\left[1-\alpha^t\gamma_w^t\right] \tag{A9}$$

and, from Eq. (A8), we have

$$\delta^{(t+1)}=\left[\left(\frac{1-\alpha}{1-\alpha\gamma_w}\right)\left(1-\alpha^t\gamma_w^t\right)+\alpha^t\gamma_w^t\right]\delta^{(1)} \tag{A10}$$

Here $\alpha^t \gamma_w^t \xrightarrow{t \to \infty} 0$ and we obtain

$$\delta^{\infty} = \left(\frac{1-\alpha}{1-\alpha\gamma_w} \right) \delta^{(1)} \tag{A11}$$

Note that $\gamma_w = 0$ means row i = row j in $\boldsymbol{W}$ and that $\alpha = 1$ yields $\delta^{(\infty)} = 0$.

Case 2. If $\alpha\gamma_w = -1$, then $\alpha = 1$, $\gamma_w = -1$, and $(1-\alpha)(1 + \alpha\gamma_w + \alpha^2\gamma_w^2 + \cdots + \alpha^{t-1}\gamma_w^{t-1}) + \alpha^t\gamma_w^t$ is 1 for even t and -1 for odd t. Hence, $\delta^{(\text{odd})} = \delta^{(1)}$ and $\delta^{(\text{even})} = -\delta^{(1)}$. Here $w_{ik} = w_{jk} = 0$ for $k \neq i,j$, and $w_{ii} = 0$, $w_{ji} = 1$, $w_{jj} = 0$, $w_{ij} = 1$. This indicates that actors i and j continually interchange their opinions.

Case 3. If $\alpha\gamma_w = 1$, then $\alpha = 1$, $\gamma_w = 1$, and $(1-\alpha)(1 + \alpha\gamma_w + \alpha^2\gamma_w^2 + \cdots + \alpha^{t-1}\gamma_w^{t-1}) + \alpha^t\gamma_w^t = (1-\alpha)t + 1 = 1$. Hence, $\delta^{(t+1)} = \delta^{(1)}$ for all t and $\delta^{(\infty)} = \delta^{(1)}$. Here $w_{ik} = w_{jk} = 0$ for $k \neq i,j$, and $w_{ii} = 1$, $w_{ji} = 0$, $w_{jj} = 1$, $w_{ij} = 0$. This indicates that there is no change of the initial opinions of actors i and j.

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