Math 35
Winter 2014
January 14

Yesterday you were trying to prove that the set of natural numbers, $\mathbb{Z}^+$, has no upper bound.

This proof uses the completeness axiom, which says that any nonempty set with an upper bound has a least upper bound.

We can use this to prove $\mathbb{Z}^+$ has no upper bound, by showing it has no least upper bound.

We can show $\mathbb{Z}^+$ has no least upper bound by assuming it does, and deriving a contradiction.

Proposition: The set $\mathbb{Z}^+$ of natural numbers has no upper bound.

Proof: We prove this by contradiction.

Suppose $\mathbb{Z}^+$ has an upper bound. By the completeness axiom, $\mathbb{Z}^+$ has a least upper bound. Let r be the least upper bound of $\mathbb{Z}^+$.

Then $r - \frac{1}{2}$ is not an upper bound for $\mathbb{Z}^+$, which means there is a natural number n such that $n > r - \frac{1}{2}$.

Adding 1 to both sides of this inequality gives us $n + 1 > r + \frac{1}{2}$. But then $n + 1$ is a natural number greater than r, contradicting the fact that r is an upper bound of $\mathbb{Z}^+$.

This completes the proof.

Another way to use the completeness axiom is to show a number with a certain property exists, by showing the least upper bound of some set X (that is nonempty and bounded above) has that property. Here is an example. Prove the following in the opposite order; that is, first prove (3), for which you may use (1) and (2). Then prove (2), and then prove (1). You may use any basic properties of $+$, $\cdot$, and $<$ that hold in ordered fields. In particular, you may use the fact that if x and y are non-negative real numbers, then $x < y$ iff $x^2 < y^2$.

1. If $a > 0$, then there is $x > 0$ such that $x^2 < a$, and there is $y > 0$ such that $y^2 > a$.

   If it helps, you may consider the cases $a < 1$, $a = 1$, and $a > 1$ separately. For $a = 1$, you can give specific values for x and y.

2. If $a > 0$ and $b > 0$, then

   (a) If $b^2 < a$, then there is $h > 0$ such that $(b + h)^2 < a$.

   (b) If $b^2 > a$, then there is $h > 0$ such that $h < b$ and $(b - h)^2 > a$.

   For (a), try setting $\varepsilon = a - b^2$, and finding h such that $(b + h)^2 - b^2 < \varepsilon$. (Note that $\varepsilon > 0$.)

3. If $a > 0$, then the completeness axiom implies that there is $b > 0$ such that $b^2 = a$.

   Hint: Consider the set $X = \{x \mid 0 < x \text{ \& } x^2 < a\}$.

For (3), consider $X = \{x \mid 0 < x \ \&\ x^2 < a\}$. By (1), there is a positive x such that $x^2 < a$, so $X \neq \emptyset$. By (1), there is a positive number y such that $y^2 > a$. Any such number y is an upper bound for X. (To see this suppose $x \in X$. Then x is positive, and $x^2 < a < y^2$. Therefore $x < y$.)

By the completeness axiom, since X is nonempty and bounded above, X has a least upper bound b. We will show that $b^2 = a$, by showing that both assumptions $b^2 < a$ and $b^2 > a$ lead to contradictions.

First suppose $b^2 < a$. By (2a), there is $b + h > b$ such that $(b + h)^2 < a$. But then $b + h \in X$, contradicting the fact that b is an upper bound for X.

Now suppose $b^2 > a$. By (2b), there is a positive $b - h < b$ such that $(b - h)^2 > a$. But we have already argued that any positive number whose square is greater than a is an upper bound for X, so this contradicts the fact that b is the *least* upper bound for X.

This completes the proof of (3).

For (2a), we have $a - b^2 = \varepsilon > 0$, and we want to find $h > 0$ such that $(b + h)^2 - b^2 < \varepsilon$. We have

$$(b + h)^2 - b^2 = 2bh + h^2 = h(2b + h).$$

If we choose h so that $h < 1$, then we will have $h(2b + h) < h(2b + 1)$. If we choose h so that $h < \frac{\varepsilon}{2b+1}$, we will have $h(2b + 1) < \varepsilon$.

Therefore, if we choose any positive $h < \min\{1, \frac{\varepsilon}{2b+1}\}$, we will have

$$(b + h)^2 - b^2 = 2bh + h^2 = h(2b + h) < h(2b + 1) < \frac{\varepsilon}{2b + 1}(2b + 1) = \varepsilon = a - b^2,$$

so (adding b^2 to both sides) $(b + h)^2 < a$.

This completes the proof of 2(a).

(2b) is similar.

For (1), notice that $(a + 1)^2 = a^2 + a + a + 1 > a$, and if $a < 1$ then (multiplying by a on both sides) $a^2 < a$. If $a \geq 1$, of course, $\left(\frac{1}{2}\right)^2 < a$.

Recall that mathematical induction is used to prove a statement of the form, "For every natural number n, $P(n)$." The proof then has two steps:

Base Case: Prove $P(1)$.

Inductive Step: Assume that n is a natural number such that $P(n)$. (This is the inductive hypothesis.) Prove $P(n+1)$.

(In other words, prove that for any natural number n, if $P(n)$, then $P(n+1)$.)

This works, because once we prove both $P(1)$ and $(P(n) \implies P(n+1))$, it follows that $P(2)$. From $P(2)$ and $(P(n) \implies P(n+1))$, it follows that $P(3)$. From $P(3)$ and $(P(n) \implies P(n+1))$, it follows that...

In this way, all the natural numbers fall into line like dominoes, and we conclude that $P(n)$ holds for every natural number n.

For example, a homework problem asks us to prove by induction that the set of natural numbers is closed under addition. To set up the proof, we follow these steps:

1. Reword the proposition so that the thing to be proved is of the form "for every natural number n, $P(n)$."

2. State that we will prove this by induction (or, if there are several variables in the statement of the proposition, "by induction on n").

3. State the base case. (say what we need to prove.

4. State the inductive step. (Say what we assume as inductive hypothesis, and what we need to prove.)

Then, once the proof is set up, we can fill in the proofs. (Yes, that's the hard part.)

Warning: For this homework problem, to prove something is a natural number, you must use the fact that the natural numbers are closed under adding 1. That is, to prove a is a natural number, you must write a in the form $b+1$ for some b that is already known (proven or assumed) to be a natural number.

In particular, it is not sufficient to prove $a > 0$. (All natural numbers are greater than zero, but so are many other real numbers.) Nor is it sufficient to give an intuitive argument about "fractional parts." You *must* use induction, and the fact that the natural numbers are closed under adding 1.

Example: Prove that every finite set of real numbers is bounded.

Proposition: For every natural number n, every set of real numbers with exactly n elements is bounded.

Proof: We prove this by induction on n.

Base Case: We must prove that every set of real numbers with exactly 1 element is bounded.

To prove this, suppose that $X = \{x_1\}$. To show X is bounded, we must find a number M such that $|x| \leq M$ for all $x \in X$. Clearly $M = |x_1|$ works.

Inductive Step: Assume that every set of real numbers with exactly n elements is bounded. This is the inductive hypothesis.

We must prove that every set of real numbers with exactly $n+1$ elements is bounded.

To prove this, suppose that $X = \{x_1, x_2, \ldots, x_{n+1}\}$. To show X is bounded, we must find a number M such that $|x| \leq M$ for all $x \in X$.

Let $Y = \{x_1, x_2, \ldots, x_n\}$. By inductive hypothesis, Y is bounded. That is, there is some number P such that $|x| \leq P$ for all $x \in Y$.

Let $M = \max\{P, |x_{n+1}|\}$. To show M works, suppose that $x \in X$. Either $x \in Y$, in which case $|x| \leq P \leq M$, or $x = x_{n+1}$, in which case $|x| = |x_{n+1}| \leq M$. This shows that $|x| \leq M$ for all $x \in X$, which shows that X is bounded.

This completes the proof.

Notes:

1. The underlined portion of the statement of the proposition is my $P(n)$. Notice that I repeated *the exact same wording* (except that n is sometimes replaced by 1, and sometimes by $n+1$) in all the underlined portions of the proof. This can really help organize your proof.

2. Appropriate notation can make life easier. For example, denoting a set of $n+1$ elements by $\{x_1, x_2, \ldots, x_{n+1}\}$ makes the main idea of the inductive step much easier to see.

3. Notice the phrases that help the reader follow the logic: "we must show ...," "to show ..., we must ...," "to show ..., suppose ...," "this shows that ...," "by inductive hypothesis, ...," 'that is, ...," and so on. Words like "because" and "therefore" are extremely useful.

4. Saying both "To show X is bounded, we must find a number M such that $|x| \leq M$ for all $x \in X$," at the beginning of the inductive step, and "This shows that $|x| \leq M$ for all $x \in X$, which shows that X is bounded," at the end of the inductive step, might seem unnecessarily repetitious. Being repetitious in order to make a proof easier to follow is not only allowed, it's encouraged.