Math 35
Winter 2014
Wednesday, February 19, Sample Solutions

Exercise 1: Explain what is wrong with the following things that could be written by calculus students confused about limits.

1. $\lim_{x\to 3} \frac{x^2-9}{x-3} = x+3 = 3+3$ so $\lim_{x\to 3} = 6$.
2. $\lim_{x\to\infty} \frac{x}{x+1} \approx 1$.
3. $\lim_{x\to 0}\left(x\sin\left(\frac{1}{x}\right)\right) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\sin\left(\frac{1}{x}\right)\right) = 0\left(\lim_{x\to 0}\sin\left(\frac{1}{x}\right)\right) = 0$.
4. $\lim_{x\to c} f(x) = L$ means that $f(x)$ gets close to L but never equals L.

Solution:

1. It makes no sense to say $\lim_{x\to 3} \frac{x^2-9}{x-3} = x+3$, since the expression on the left of the equals sign denotes a number, and the expression on the right denotes a function of x.

   It also makes no sense to say $\lim_{x\to 3} = 6$, since the expression on the left of the equals sign doesn't denote anything at all.

   This should read
   $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6$.

2. $\lim_{x\to\infty} \frac{x}{x+1}$ is a number, and it is not approximately equal to 1, it *is* equal to 1. This should read $\lim_{x\to\infty} \frac{x}{x+1} = 1$.

   However, we could say, somewhat loosely, that for very large values of x we have $\frac{x}{x+1} \approx 1$.

3. The theorem that "the limit of the product is the product of the limits" can be applied only when the relevant limits (the limits of the factors) exist. We could apply exactly the same reasoning with $\sin\left(\frac{1}{x}\right)$ replaced by $\frac{1}{x}$ to get
$\lim_{x\to 0}\left(x\left(\frac{1}{x}\right)\right) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\left(\frac{1}{x}\right)\right) = 0\left(\lim_{x\to 0}\left(\frac{1}{x}\right)\right) = 0,$
which is clearly wrong.

   Of course, it is true that $\lim_{x\to 0}\left(x\sin\left(\frac{1}{x}\right)\right) = 0$. We proved this using the squeeze theorem.

4. It is not correct to say "$f(x)$ never equals L." We can have $\lim_{x\to c} f(x) = L$ and also have $f(x) = L$ for some (or many, or all) values of x near c. As just one example, for $f(x) = x\left(\sin\frac{1}{x}\right)$, we have $\lim_{x\to 0} f(x) = 0$, and every open interval containing 0 contains infinitely many points x for which $f(x) = 0$.

Exercise 2: We saw that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational;} \end{cases}$$

is not continuous at any point.

Define a function that is continuous at 0 but nowhere else.

Define a function that is discontinuous at every point $\frac{1}{n}$ (where n is a natural number), but continuous everywhere else (including at 0).

Solution:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational;} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x = \frac{1}{n} \text{ for some natural number } n; \\ -x & \text{otherwise.} \end{cases}$$

Exercise 3: Suppose that f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x)$ exists. (Note, this means that $\lim_{x\to\infty} f(x)$ is a finite number.) Show that f is uniformly continuous on $[0,\infty)$.

Recall, this means that

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x, y \in [0,\infty))\big(|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon\big),$$

where δ may depend on ε but not on x or y.

We have seen that if a function is continuous on a closed, bounded interval $[a, b]$, then it is uniformly continuous there. This may be useful.

Solution:

Suppose $\lim_{x\to\infty} f(x) = L$.

Let $\varepsilon > 0$. Because $\lim_{x\to\infty} f(x) = L$, there is some b such that, for all $x > b$, we have $|f(x) - L| < \frac{\varepsilon}{2}$.

Because f is uniformly continuous on the closed interval $[0, b+1]$, there is some $\delta > 0$ such that, for all $x, y \in [0, b+1]$, we have

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

We may safely assume $\delta \le 1$.

Now, suppose that $x, y \in [0,\infty)$ and $|x - y| < \delta$. Since $|x - y| < 1$, either both x and y are less than $b + 1$, or both x and y are greater than b. In the first case, x and y are both in $[0, b + 1]$, and $|x - y| < \delta$, so by our choice of δ, we have $|f(x) - f(y)| < \varepsilon$. In the second case, both x and y are greater than b, so by our choice of b, we have $|f(x) - L| < \frac{\varepsilon}{2}$ and $|f(y) - L| < \frac{\varepsilon}{2}$, so it follows that $|f(x) - f(y)| < \varepsilon$.

Putting the two cases together, we see that if $x, y \in [0,\infty)$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Since for any $\varepsilon > 0$ we can choose such a $\delta > 0$, this shows f is uniformly continuous on $[0,\infty)$.

Exercise 4: Monday's exercises culminated in an example showing that f can be continuous on a closed, bounded interval $[a, b]$, but not of bounded variation on $[a.b]$.

Show that if f is continuous on $[a, b]$, and its derivative f' is also continuous on $[a, b]$, then f is of bounded variation on $[a, b]$. You may use basic facts about derivatives from calculus, including the Mean Value Theorem:

If f is differentiable on $[c,d]$, then there is a point $z \in (c,d)$ such that $f'(z) = \dfrac{f(d) - f(c)}{d - c}$.

Solution:

Suppose f and $[a,b]$ are as specified. Because the derivative of f is continuous on $[a,b]$, it is also (by the Extreme Value Theorem) bounded on $[a,b]$. Let M be a bound; that is, for all $c \in [a,b]$, we have $|f'(c)| < M$.

Suppose $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is any partition of $[a,b]$. For any i with $1 \leq i \leq n$, by the Mean Value Theorem, we have a point $z \in [x_{i-1}, x_i]$ such that $f'(z) = \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$.

Since $|f'(z)| < M$, we have $\left| \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right| < M$, or

$$|f(x_i) - f(x_{i-1}| < M|x_i - x_{i-1}| = M(x_i - x_{i-1}).$$

From this, we see that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \sum_{i=1}^{n} M(x_i - x_{i-1}) = M \sum_{i=1}^{n} (x_i - x_{i-1}) = M(x_n - x_0) = M(b-a).$$

Since $V(f,[a,b])$ is the supremum of the set of all such sums, this shows that $V(f,[a,b]) \leq M$.

In particular, f is of bounded variation on $[a.b]$.