Math 35
Winter 2014
Convergence of Sequences: Example Proofs

Proposition: If the sequence $\{a_n\}$ converges to the real number a and the sequence $\{b_n\}$ converges to the real number b, then the sequence $\{a_n + b_n\}$ converges to the real number $a + b$.

Proof: Suppose that $\{a_n\}$ converges to a and $\{b_n\}$ converges to b. We must show $\{a_n + b_n\}$ converges to $a + b$.

To show this, let $\varepsilon > 0$. We must show there is N such that, for all $n \geq N$, we have $|(a_n + b_n) - (a + b)| < \varepsilon$.

Because $\{a_n\}$ converges to a, there is a number N_a such that, for all $n \geq N_a$, we have $|a_n - a| < \frac{\varepsilon}{2}$.

Because $\{b_n\}$ converges to b, there is a number N_b such that, for all $n \geq N_b$, we have $|b_n - b| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_a, N_b\}$.

To show this works, suppose that $n \geq N$. We must show that we have $|(a_n + b_n) - (a + b)| < \varepsilon$.

Since $n \geq N \geq N_a$, we have $|a_n - a| < \frac{\varepsilon}{2}$, and similarly, we also have $|b_n - b| < \frac{\varepsilon}{2}$. Using the Triangle Inequality, we have

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This is what we needed to show. $\square$

Proposition: Suppose the sequence $\{b_n\}$ converges to a positive number b. Then $\{\frac{1}{b_n}\}$ converges to $\frac{1}{b}$.

Proof: Let $\varepsilon > 0$ be given. Define

$$h = min\left\{\frac{b}{2}, \frac{\varepsilon b^2}{4}\right\},$$

and choose N such that, for all $n \geq N$, we have $|b_n - b| < h$. We must show that, for all $n \geq N$, we have $|\frac{1}{b_n} - \frac{1}{b}| < \varepsilon$.

Since $h \leq \frac{b}{2}$ we have $0 < b - h < b < b + h$, and so $\frac{1}{b+h} < \frac{1}{b} < \frac{1}{b-h}$. For $n \geq N$, we also have $0 < b - h < b_n < b + h$, and so $\frac{1}{b+h} < \frac{1}{b_n} < \frac{1}{b-h}$. Therefore, since $\frac{1}{b_n}$ and $\frac{1}{b}$ both lie between $\frac{1}{b+h}$ and $\frac{1}{b-h}$, we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \left|\frac{1}{b-h} - \frac{1}{b+h}\right|,$$

and if we can show that $\left|\frac{1}{b-h} - \frac{1}{b+h}\right| \leq \varepsilon$, we will be done.

Since $h \leq \frac{b}{2}$, we have $b - h \geq \frac{b}{2}$ and $b + h > b$, so $(b-h)(b+h) > \frac{b^2}{2}$, and

$$\frac{1}{(b-h)(b+h)} < \frac{2}{b^2}.$$

We will use this in the following calculation.

$$\frac{1}{b-h} - \frac{1}{b+h} = \frac{2h}{(b-h)(b+h)} < 2h\left(\frac{2}{b^2}\right) = h\left(\frac{4}{b^2}\right) \leq \left(\frac{\varepsilon b^2}{4}\right)\left(\frac{4}{b^2}\right) = \varepsilon.$$

This is what we needed to show. □

Proposition: Suppose $\{b_n\}$ is a sequence of nonzero numbers that converges to 0. Then $\{\frac{1}{b_n}\}$ diverges.

Proof: Let L be any real number. To show $\{\frac{1}{b_n}\}$ does not converge to L, set $\varepsilon = 1$, and let N be given. We must show there is $n \geq N$ such that

$$\left|\frac{1}{b_n} - L\right| \geq 1.$$

Choose M such that, for all $n \geq M$, we have $|b_n - 0| < \dfrac{1}{|L|+1}$, and choose any n greater than $max\{N, M\}$. Then we have $|b_n| < \dfrac{1}{|L|+1}$, and so

$$\left|\frac{1}{b_n}\right| > |L| + 1.$$

Using the Reverse Triangle Inequality,

$$\left|\frac{1}{b_n} - L\right| \geq \left|\left|\frac{1}{b_n}\right| - |L|\right| > 1.$$

This is what we needed to show. □