Math 35
Winter 2014
Axioms for the Real Numbers

These axioms are as phrased in your textbook.

Definition 1.1 A **field** is a nonempty set F of objects that has two operations defined on it. These operations are called addition and multiplication and are denoted in the usual way. Addition and multiplication satisfy the following properties:

1. $x + y \in F$ for all $x, y \in F$.
2. $x + y = y + x$ for all $x, y \in F$.
3. $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
4. F contains an element 0 such that $x + 0 = x$ for all $x \in F$.
5. For each $x \in F$ there exists $y \in F$ such that $x + y = 0$.
6. $xy \in F$ for all $x, y \in F$.
7. $xy = yx$ for all $x, y \in F$.
8. $(xy)z = x(yz)$ for all $x, y, z \in F$.
9. F contains an element 1 such that $x \cdot 1 = x$ for all $x \in F$.
10. For each $x \in F$ such that $x \neq 0$ there exists $y \in F$ such that $xy = 1$.
11. $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Definition 1.2 An **order** on a set S is a relation that satisfies the following two properties:

1. If $x, y \in S$ then exactly one of $x < y$, $x = y$, or $y < x$ is true.
2. For all $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

An **ordered set** is a set with an order defined on it.

Definition 1.3 An **ordered field** is a field F that is an ordered set with the following additional properties:

1. If $x > 0$ and $y > 0$ then $x + y > 0$.
2. If $x > 0$ and $y > 0$ then $xy > 0$.
3. $x < y$ if and only if $y - x > 0$.

$\mathbb{R}$ is an ordered field that also satisfies the Completeness Axiom:

Definition 1.14(a) Let S be a nonempty set of real numbers. The set S is **bounded above** if there is a number M such that $x \leq M$ for all $x \in S$. The number M is called an **upper bound** of S.

Definition 1.15(a) Let S be a nonempty set of real numbers. Suppose S is bounded above. The number β is the **supremum** of S if β is an upper bound of S and any number less than β is not an upper bound of S. We will write $\beta = \sup S$.

Completeness Axiom: Each nonempty set of real numbers that is bounded above has a supremum.