INTEGRAL TRANSFORMS.

ABSTRACT. This is a reference guide to the basic theory of the Laplace and Fourier transforms and their application to ODEs.

1. Introduction

An integral transform is an operator, i.e. a map from functions to functions that takes the form

$$I(f)(\xi) = \int_{-\infty}^{\infty} K(x,\xi) f(x)dx.$$

The function of two-variables K is called the kernel of the transform. In general, a lot of the properties of the transform, such as for which functions f the integral even makes sense, depend on K, but all integral transforms have the following linearity properties

- $I(f+g) = I(f) + I(g)$
- $I(cf) = cI(f)$ for constants c.

2. The Laplace Transform

The Laplace transform is the integral transform with kernel

$$K(x,\xi) = \chi_{[0,\infty)}(x)e^{-x\xi}.$$

Because the kernel is only non-zero for positive x, it is traditional to think of the variable x as time and relabel it t. Likewise it is traditional to use s rather than ξ here.

Equivalently we can make the definition

Definition 2.1. *The Laplace transform of a function $f(t)$ is the function*

$$\mathcal{L}\left[f\right](s) = \int_0^{\infty} f(t)e^{-st}dt$$

Because the kernel decays rapidly, the Laplace transform makes sense for most functions, in fact anything that grows exponentially fast or slower. We'd need to consider a functions growing as rapidly as e^{t^2} or $e^{t\log t}$ to have difficulties.

Tables of standard Laplace transforms can be found many places, e.g. Logan p.203.

Lemma 2.2. *If $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0$ is a polynomial with constant constant coefficients then for $P = p(D)$,*

$$\mathcal{L}\left[p(D)y\right](s) = p(s)\mathcal{L}\left[y\right](s) - s^{n-1}y(0) - s^{k-2}y'(0) - \cdots - y^{(k-1)}(0).$$

The Laplace transform allows us to reduce the solution of inhomogeneous constant coefficient linear ODEs to an algebraic exercise in partial fractions via the following theorem. This is especially useful for forward-time only equations (i.e. only for $t \geq 0$) as we can also normalize initial conditions.

Theorem 2.3. *Suppose that p is a polynomial as before then there exists a C^∞ function g such that*

$$\mathcal{L}\left[g\right](s) = \frac{1}{p(s)}.$$

In addition, $u(t) = \chi_{[0,\infty)}(t)g(t)$ is then a fundamental solution for the operator $P = p(D)$.

Furthermore for any piecewise continuous differentiable function $f(t)$ (with finite jumps) the convolution $y = u * (\chi_{[0,\infty)}f)$ is continuous and defined for all time and solves

$$Py = \chi_{[0,\infty)}f$$

together with the initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$

Remark 2.4. *In fact, it can be shown that the function $g(t)$ is the unique solution to the homogenous IVP, $Py = 0$, $y(0) = \cdots = y^{(k-2)}(0) = 0$, $y^{(k-1)}(0) = 1$.*

Example 2.5. Solve the initial value problem

$$y''' + y = \chi_{[0,\infty)}(t)\log(t), \quad y(0) = 0, y'(0) = 0, y''(0) = 0.$$

Here we must solve $\mathcal{L}\left[g\right](s) = \frac{1}{s^3+1}$ for a smooth function g. Now using partial fractions

$$\begin{aligned}\frac{1}{1+s^3} &= \frac{1}{(s+1)(s^2-s+1)} = \frac{1/3}{s+1} + \frac{2/3 - s/3}{s^2-s+1} \\ &= \frac{1}{3}\left(\frac{1}{s+1} - \frac{s-1/2}{(s-1/2)^2+3/4}\right) + \frac{1/2}{(s-1/2)^2+3/4}\end{aligned}$$

Applying the inverse Laplace transform we see

$$g(t) = \frac{1}{3}\left(e^{-t} - e^{t/2}\cos\left(\frac{t\sqrt{3}}{2}\right)\right) + \frac{1}{\sqrt{3}}e^{t/2}\sin\left(\frac{t\sqrt{3}}{2}\right).$$

The solution to the IVP is then

$$y(t) = \int_0^t g(\tau)\log(t-\tau)d\tau.$$

□

Example 2.6. To solve with other initial conditions, e.g.

$$y''' + y = \chi_{[0,\infty)}(t)\log(t), \quad y(0) = 0, y'(0) = 1, y''(0) = 0,$$

solve the homogeneous problem $y''' + y = 0$ with these initial conditions by elementary methods and then add the solution from the previous example. □

3. The Fourier Transform

The Fourier transform defined by

$$\mathcal{F}\left[u\right](\xi) = \hat{u}(\xi) = \int_{-\infty}^{\infty} u(x)e^{ix\xi}dx$$

is an integral transform much like the Laplace transform. However its kernel

$$K(x,\xi) = e^{ix\xi} = cos(x\xi) + i\sin(x\xi)$$

is oscillatory and does not decay. Its theory is more complicated, but much richer than that of the Laplace transform. Since $|K(x,\xi)| \leq 1$ for all x, ξ, the Fourier transform makes sense (as a function) for $f \in L^1(\mathbb{R})$, but not for any periodic or increasing functions.

Remark 3.1. *It's not easy to prove, but the Fourier transform does make sense for functions $f \in L^2(\mathbb{R})$. Furthermore it can be shown that $\hat{u} \in L^2(\mathbb{R})$ and $\|\hat{u}\|_{L^2(\mathbb{R})} = \|u\|_{L^2(\mathbb{R})}$. Although we shall not need this during this course, it is an example of a powerful result in advanced Fourier analysis.*

We would like to make sense of the Fourier transform for a much larger class of functions than purely those in $L^1(\mathbb{R})$ (or L^2). To do this we can try to think of $\hat{u}$ as a distribution and define

$$\begin{aligned}
\hat{u}(\xi)(\phi) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(x)e^{ix\xi}dx\right)\phi(\xi)d\xi \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{ix\xi}\phi(\xi)d\xi\right)u(x)dx \\
&= \int_{-\infty}^{\infty} u(x)\hat{\phi}(x)dx \\
&= u(x)(\hat{\phi}).
\end{aligned} \tag{1}$$

Unfortunately, there is a problem with this idea. For a test function ϕ, the Fourier transform $\hat{\phi}$ may not be a test function. Thus we might not be able to make sense of the final line of (1).

Remark 3.2. *In fact, if both ϕ and $\hat{\phi}$ are test functions, it is possible to show that $\phi = 0$ everywhere. This is a surprisingly deep result, that has connections with many fields. It is phrased as "you cannot simultaneously localize in both space and frequency" and thus can philosophically be thought of as a mathematical formulation of the Heisenberg uncertainty principle.*

Because of this limitation, we introduce a class of 'almost' test functions.

Definition 3.3. *The Schwartz class $\mathcal{S}$ is the collection of all C^∞ functions $f(x)$ that have the property:*

The product $x^\alpha f^{(n)}(x)$ is bounded for all $\alpha \geq 0$ and integers $n \geq 0$.

Remark 3.4. *This means that f and all of its decay more rapidly than any polynomial as $|x| \to \infty$. In fact the definition is equivalent to*

$$\left|x^\alpha f^{(n)}(x)\right| \to 0 \qquad \text{as } |x| \to \infty$$

for all $\alpha \geq 0$ and $n \geq 0$. In other words, any derivative of f mulitplied by any polynomial must vanish at infinity.

The Schwartz class has many nice properties

- All test functions are Schwartz functions.
- All Schwartz functions are in L^1 and so have Fourier transforms (as functions).
- If $f, g \in \mathcal{S}$ then $f + cg \in \mathcal{S}$ for all constants c.
- If $f \in \mathcal{S}$ then $f' \in \mathcal{S}$
- $p(x)f(x) \in \mathcal{S}$ for all polynomials p.
- $x^\alpha f^{(n)}(x) \in \mathcal{S}$ for all $\alpha, n \geq 0$.

Lemma 3.5. *For $u, v \in \mathcal{S}$, real constant a and polynomial p, the following properties hold:*

(a) $\mathcal{F}[u + av] = \widehat{u} + a\widehat{v}$
(b) $\mathcal{F}[u(x-a)](\xi) = e^{ia\xi}\widehat{u}(\xi)$
(c) $\mathcal{F}\left[e^{-iax}u\right](\xi) = \widehat{u}(\xi - a)$
(d) $\mathcal{F}[p(D_x)u](\xi) = p(-i\xi)\widehat{u}(\xi)$
(e) $\mathcal{F}[p(x)u(x)](\xi) = p(-iD_\xi)\widehat{u}(\xi)$
(f) $\mathcal{F}[u(ax)] = \frac{1}{a}\widehat{u}(\xi/a)$
(g) $\mathcal{F}[u * v] = \widehat{u}\widehat{v}$.

*Many of these properties (esp. (b), (c)) extend naturally to imaginary and complex constants a, but **only** if applying them does not introduce functions that have faster than polynomial growth at infinity.*

These properties are left as en exercise for the reader. They have an important corollary.

Corollary 3.6. *If $u(x) \in \mathcal{S}$ then $\widehat{u}(\xi) \in \mathcal{S}$.*

Proof: We need to show that $\xi^\alpha (D_\xi)^n \widehat{u}$ is bounded for all $\alpha, n \geq 0$. A key observation is that for any $f \in \mathcal{S}$

$$|\widehat{f}(\xi)| = \left|\int_{-\infty}^{\infty} f(x)e^{ix\xi}dx\right| \leq \int_{-\infty}^{\infty} |f(x)e^{ix\xi}|dx = \int_{-\infty}^{\infty} |f(x)|dx. \tag{2}$$

Now by Lemma 3.5

$$\xi^\alpha D_\xi^n \widehat{u} = i^n \xi^\alpha \mathcal{F}[x^n u(x)] = i^{\alpha+n}\mathcal{F}[D_x^\alpha(x^n u)].$$

But $D^\alpha(x^n u)$ is a Schwartz function so we can use (2) to see

$$\left|\xi^\alpha D_\xi^n \widehat{u}\right| \leq \int_{-\infty}^{\infty} |D_x^\alpha(x^n u(x))|\, dx$$

The integrand must decay faster than any polynomial in x, so the integral must converge to a constant.

■

The consequence of this result is that Schwartz functions are the ideal replacement for test functions when studying the Fourier transform.

Definition 3.7. *A tempered distribution is a distribution that makes sense when applied to all Schwartz functions.*

Example 3.8. The delta function and its derivatives are tempered distributions.

Any piecewise continuous function with finite jumps that grows at most like a polynomial as $|x| \to \infty$ is a tempered distribution. (This includes functions that decay towards both $\pm\infty$.)

Functions such as e^x which have exponential growth as $x \to \infty$ are **not** tempered distributions. □

Definition 3.9. *The Fourier transform of a tempered distribution T is the tempered distribution $\mathcal{F}[T] = \widehat{T}$ defined by*

$$\widehat{T}(\phi) = T(\widehat{\phi})$$

for all Schwartz function ϕ.

Remark 3.10. *All the properties of the Fourier transform from Lemma 3.5 also apply to tempered distributions! The argument is to use the definition to move the transform onto the Schwartz function, apply Lemma 3.5, and then move it back.*

The Fourier transform is invertible. A tedious computation yields the following theorem

Theorem 3.11. *For* $\widehat{u}(\xi) \in \mathcal{S}$

$$\mathcal{F}^{-1}\left[\widehat{u}\right](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(\xi)e^{-ix\xi}d\xi.$$

This extends to tempered distributions by

$$\mathcal{F}^{-1}\left[T\right](\phi) = T(\mathcal{F}^{-1}\left[\phi\right]).$$

The Fourier transform and its inverse are closely related by the following formula

Corollary 3.12.

$$\widehat{u}(\xi) = 2\pi\mathcal{F}^{-1}\left[u\right](-\xi)$$

and hence

$$\widehat{\widehat{u}}(x) = -2\pi u(-x).$$

Proof: Exercise for the reader

■

Of course, all this means that many Fourier transforms are difficult to compute. We'll compute some here and a table can be found at the end of these notes.

Example 3.13. The Fourier transform of the delta function:

Formally, using the definitions

$$\begin{aligned}\mathcal{F}\left[\delta(x-a)\right](\phi) = \delta(x-a)(\widehat{\phi}(x)) &= \delta(x-a)\left(\int_{-\infty}^{\infty}\phi(\xi)e^{ix\xi}d\xi\right)\\ &= \int_{-\infty}^{\infty}\phi(\xi)e^{ia\xi}d\xi\end{aligned}$$

Thus as a distribution

$$\delta\widehat{(x-a)}(\xi) = e^{ia\xi} = \cos(a\xi) + i\sin(a\xi).$$

Informally, if we pretend $\delta(x-a)$ is a function and try to take the Fourier transform directly we see

$$\delta\widehat{(x-a)}(\xi) = \int_{-\infty}^{\infty}\delta(x-a)e^{ix\xi}dx = e^{ia\xi}.$$

□

Example 3.14. The Fourier transform of $\sin x$: Note $\sin x$ is not Schwartz (or L^2), so the Fourier transform should be interpreted as a distribution.

Using the fact that $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$, (which follows from $e^{i\theta} = \cos\theta + i\sin\theta$), we see that

$$\sin\xi = \frac{1}{2i}\left(\delta\widehat{(x-1)} - \delta\widehat{(x+1)}\right).$$

Thus we deduce that

$$\mathcal{F}^{-1}\left[\sin\xi\right] = \frac{1}{2i}\left(\delta(x-1) - \delta(x+1)\right)$$

Using Corollary 3.12 we see that

$$\mathcal{F}\left[\sin x\right](\xi) = \frac{\pi}{i}\left(\delta(\xi-1)-\delta(\xi+1)\right) = \pi i\left(\delta(\xi+1)-\delta(\xi-1)\right).$$

□

Example 3.15. We can sometimes push things even a little further and take Fourier transforms of functions that technically aren't even tempered distributions. For example $u(x) = \frac{1}{x}$ has an infinite jump discontinuity at 0, so isn't even a distribution. However applying Corollary 3.12

$$\mathcal{F}\left[xu(x)\right] = \mathcal{F}\left[1\right] = \mathcal{F}\left[\mathcal{F}\left[\delta\right]\right] = -2\pi\delta(-\xi) = 2\pi\delta(\xi).$$

(Ex: show that $\delta(ax) = \frac{1}{a}\delta(x)$). But by Lemma 3.5

$$\mathcal{F}\left[xu(x)\right] = -iD_\xi\widehat{u}$$

and so

$$\mathcal{F}\left[1/x\right](\xi) = C - 2\pi i\chi_{[0,\infty)}(\xi)$$

for some constant C.

In fact (by computing the inverse Fourier transform of the rhs) , it can be shown that $C = \pi i$ and so

$$\mathcal{F}\left[1/x\right](\xi) = \pi i\chi_{[0,\infty)}(-\xi) - \pi i\chi_{[0,\infty)}(\xi).$$

The most important consequence of this is that

$$\mathcal{F}^{-1}\left[\frac{1}{\xi-a}\right] = \frac{1}{2}\left(\sin(ax) - i\cos(ax)\right)\left(\chi_{[0,\infty)}(-x) - \chi_{[0,\infty)}(x)\right)$$

The three cases are due to the need to avoid exponential growth in various integrands. □

Example 3.16. Find a fundamental solution for the operator $p(D) = D^2 + 2D + 2$:

By Lemma 3.5,

$$\mathcal{F}\left[p(D)u\right] = p(-i\xi)\widehat{u} = (-\xi^2 - 2i\xi + 2)\widehat{u} = -(\xi+1+i)(\xi-1+i)\widehat{u}.$$

Now $\widehat{\delta} = 1$, so if we Fourier transform both sides of the equation

$$p(D)u = \delta$$

we get

$$\widehat{u} = -\frac{1}{(\xi+1+i)(\xi-1+i)} = \frac{1/2}{\xi+1+i} - \frac{1/2}{\xi-1+i}.$$

Thus by Example 3.15,

$$\begin{aligned} u(x) &= \frac{1}{4}\left(\chi_{[0,\infty)}(-x) - \chi_{[0,\infty)}(x)\right)e^{-x}\left(\sin(-x) - i\cos(-x) - \sin(x) + i\cos(x)\right) \\ &= \frac{1}{2}\left(\chi_{[0,\infty)}(x) - \chi_{[0,\infty)}(-x)\right)e^{-x}\sin(x). \end{aligned}$$

□

Remark 3.17. *Since every polynomial factors into linear terms over the complex field, partial fractions is theoretically much simpler for the Fourier transform than the Laplace transform. However, for practical problems it is often better to apply short cuts. For example, the inverse Fourier transform of* $\frac{1}{a^2+\xi^2}$ *can be computed (Ex) without using Example 3.15. This provides an alternative starting point for inverting the Fourier transform to compute a fundamental solution.*

Remark 3.18. *Fundamental solutions computed using the Fourier transform do not have the same initial value properties as those computed using the Laplace transform. However, for differential equations on the whole spatial domain, they often have global symmetry properties that make them theoretically more pleasant to use. Generally speaking, we use the Laplace transform with time, and the Fourier transform with space.*

Table of Fourier Transforms

$f(x)$	$\mathcal{F}[f](\xi)$
$u(x)$	$\widehat{u}(\xi)$
$p(x)u(x)$	$p(-iD_\xi)\widehat{u}(\xi)$
$p(D)u(x)$	$p(-i\xi)\widehat{u}(\xi)$
$u(x-a)$	$e^{ia\xi}\widehat{u}(\xi)$
$e^{-iax}u(x)$	$\widehat{u}(\xi-a)$
$u*v(x)$	$\widehat{u}(\xi)\widehat{v}(\xi)$
$u(ax)$	$\frac{1}{a}\widehat{u}(\xi/a)$
$\widehat{u}(x)$	$-2\pi u(-\xi)$
$\delta(x-a)$	$e^{ia\xi}$
e^{iax}	$2\pi\delta(\xi-a)$
$\sin x$	$\pi i\left(\delta(\xi+1)-\delta(\xi-1)\right)$
$\cos x$	$\pi\left(\delta(\xi+1)+\delta(\xi-1)\right)$
$\frac{1}{2}\left(\delta(x-a)-\delta(x+a)\right)$	$\sin(a\xi)$
$\frac{1}{2}\left(\delta(x-a)+\delta(x+a)\right)$	$\cos(a\xi)$
x^n, $n=0,1,\ldots$	$(-i)^n 2\pi\delta^{(n)}(\xi)$
$\frac{1}{2a}e^{-a\|x\|}$	$\frac{1}{a^2+\xi^2}$
$\frac{1}{2}(\chi_{[0,\infty)}(-x)-\chi_{[0,\infty)}(x))\sin(x)$	$\frac{1}{\xi^2-a^2}$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{\xi^2}{4a}}$
$\mathrm{sinc}(x)=\frac{\sin x}{x}$	$\pi\chi_{[-1,1]}(\xi)$
$\frac{1}{2a}\chi_{[-a,a]}(x)$	$\mathrm{sinc}(a\xi)=\frac{\sin(a\xi)}{a\xi}$
$\mathrm{sinc}^2(x)$	$\frac{\pi}{2}\chi_{[-2,2]}(\xi)(2-\|\xi\|)$
$\frac{1}{2}\left(\sin(ax)-i\cos(ax)\right)\left(\chi_{[0,\infty)}(-x)-\chi_{[0,\infty)}(x)\right)$	$\frac{1}{\xi-a}$
$-ie^{-bx}e^{-iax}\chi_{[0,\infty)}(x), \quad b>0$	$\frac{1}{\xi-a+ib}$
$ie^{bx}e^{-i(ax}\chi_{[0,\infty)}(-x), \quad b>0$	$\frac{1}{\xi-a-ib}$