Major Facts about Rings

Axioms of Rings

A nonempty set R equipped with two binary operations, addition and multiplication, is called **a ring** **<u>if</u>** for all $a, b, c \in R$:

1. $a + b = b + a$;
2. $(a + b) + c = a + (b + c)$;
3. there is $0 \in R$ such that $a + 0 = a$ for all $a \in R$ (additive identity);
4. there is $-a \in R$ such that $a + (-a) = 0$;
5. $a(bc) = (ab)c$;
6. $a(b + c) = ab + bc$ and $(b + c)a = ba + ca$.

Fact 1. **(Rules of multiplication)** Let R be a ring and $a, b, c \in R$. **<u>Then</u>**

1. $a0 = 0a = 0$;
2. $a(-b) = (-a)b = -(ab)$;
3. $(-a)(-b) = ab$;
4. $a(b - c) = ab - ac$ and $(b - c)a = ba - ca$.

   Furthermore, if R and a **unity** element 1, **<u>then</u>**

5. $(-1)a = -a$;
6. $(-1)(-1) = 1$.

Fact 2. **(Uniqueness of the Unity and Inverses)**

If a ring has a unity, it is **unique**. If a ring element has a multiplicative inverse, it is **unique**.

Fact 3. **(Subring Test)**

Let $S \in R$ by a subset of a ring R. **<u>Then</u>** S is a subring **<u>if and only if</u>** S is closed under subtraction and multiplication, that is, $a - b \in S$ and $ab \in S$ for all $a, b \in S$.