(Ir)Reducibility Tests for Polynomials

Test 1. **(Reducibility Test for Degrees 2 and 3)**
Let F be a field, $f(x) \in F[x]$, and $\deg f(x) = 2$ or 3. **Then** $f(x)$ is reducible over F **if and only if** $f(x)$ has a zero in F.

Test 2. **($\mathbb{Q}$ implies $\mathbb{Z}$)**
Let $f(x) \in \mathbb{Z}[x]$. **If** $f(x)$ is reducible over $\mathbb{Q}$, **then** it is also reducible over $\mathbb{Z}$.

Test 3.
Let p be a prime number and $f(x) \in \mathbb{Z}[x]$ with $\deg f(x) \geq 1$. Let $\overline{f}(x) \in \mathbb{Z}_p[x]$ be obtained from $f(x)$ by reducing all the coefficients modulo p. **If** $\overline{f}(x)$ is irreducible over $\mathbb{Z}_p$ and $\deg \overline{f}(x) = \deg f(x)$, **then** $f(x)$ is irreducible over $\mathbb{Q}$.

Test 4. **(Eisenstein's Criterion)**
Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$. **If** there is a prime p such that $p \mid a_{n-1}$, $p \mid a_{n-2}$, $\ldots$, $p \mid a_0$, **but** $p \nmid a_n$ and $p^2 \nmid a_0$, **then** $f(x)$ is irreducible over $\mathbb{Q}$.

Corollary 4.1. **(Irreducibility of Cyclotomic Polynomials)**
Let $\Phi_p(x) = \dfrac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$ be the p-th cyclotomic polynomial for $p \geq 2$. **Then** $\Phi_p(x)$ is irreducible over $\mathbb{Q}$ **if and only if** p is prime.