Major Facts about Isomorhpisms

Fact 1. **(Cayley's Theorem)** Every group is isomorphic to a group of permutations.

Fact 2. **(Action on Group Elements)** Let $\varphi : G \to \overline{G}$ be an isomorphism. **Then**

a. φ maps the identity of G into the identity of $\overline{G}$, that is, $\varphi(e) = \overline{e}$;

b. for every $a \in G$ and every integer n, $\varphi(a^n) = (\varphi(a))^n$;

c. $a, b \in G$ commute **if and only if** $\varphi(a)$ and $\varphi(b)$ commute;

d. $|a| = |\varphi(a)|$ for all $a \in G$;

e. fix an integer k and a group element $b \in G$. **Then** the equation $x^k = b$ has the same number of solutions in G, as does the equation $\overline{x}^k = \varphi(b)$ in $\overline{G}$.

Fact 3. **(Action on Groups)** Let $\varphi : G \to \overline{G}$ be an isomorphism. **Then**

a. G is Abelian **if and only if** $\overline{G}$ is Abelian;

b. G is cyclic **if and only if** $\overline{G}$ is cyclic;

c. φ^{-1} is an isomorphism from $\overline{G}$ onto G;

d. **if** H is a subgroup of G, **then** $\varphi(H) = \{\varphi(h) \mid h \in H\}$ is a subgroup of $\overline{G}$.