Major Facts about Homomorhpisms

Fact 1. **(Properties on Group Elements)** Let $\varphi : G \to \overline{G}$ be a homomorphism. **Then**

a. φ maps the identity of G into the identity of $\overline{G}$, that is, $\varphi(e) = \overline{e}$;

b. for every $g \in G$ and every integer n, $\varphi(g^n) = (\varphi(g))^n$;

c. $|\varphi(g)|$ divides $|g|$ for all $g \in G$ such that $|g| < \infty$;

d. $\ker\varphi$ is a (normal) subgroup of G;

e. **if** $\varphi(g) = g' \in \overline{G}$, **then** $\varphi^{-1}(g') = g\ker\varphi$, where $\varphi^{-1}(g') = \{x \in G \mid \varphi(x) = g'\}$.

Fact 2. **(Properties of Subgroups)** Let $\varphi : G \to \overline{G}$ be an homomorphism, $H \leq G$, and $\overline{K} \leq \overline{G}$. **Then**

a. $\varphi(H) = \{\varphi(h) \mid h \in H\}$ is a subgroup of $\overline{G}$;

b. **if** H is cyclic, **then** $\varphi(H)$ is cyclic;

c. **if** H is Abelian, **then** $\varphi(H)$ is Abelian;

d. **if** H is normal in G, **then** $\varphi(H)$ is normal in $\overline{G}$;

e. **if** $|\ker\varphi| = n$, **then** φ is an n-to-1 mapping from G **onto** $\varphi(G)$;

f. **if** $|H| = n$, then $|\varphi(H)|$ divides n;

g. $\varphi^{-1}(\overline{K}) = \{k \in G \mid \varphi(k) \in \overline{K}\}$ is a subgroup of G;

h. **if** $\overline{K}$ is a normal subgroup in $\overline{G}$, **then** $\varphi^{-1}(\overline{K})$ is a normal subgroup of G;

i. **if** φ is onto **and** $\ker\varphi = \{e\}$, **then** φ is an isomorphism from G to $\overline{G}$.

Fact 3. **(First Isomorphism Theorem)** Let $\varphi : G \to \overline{G}$ be a homomorphism.
Then $G/\ker\varphi \approx \varphi(G)$ and the isomorphism is given by $g\ker\varphi \mapsto \varphi(g)$.

Fact 4. **(Normal Subgroups are Kernels)** Every normal subgroup N of a group G is a kernel of the homomorphism from G to G/N given by $g \mapsto gN$.