Math 31: Abstract Algebra
Fall 2018 - Essay

Return date: Thursday 11/15/18 at 4 pm in KH 318

Instructions: Write your answers neatly and clearly, use complete sentences and label any diagrams. If you shall prove something, then show your work.

Introduction

Write a short introduction about the 2x2 cube and/or the 3x3 cube in general.

Questions

Let P_0 be the cube in the solved position with fixed sides (i.e. front side = yellow, up side = green, left side = orange etc.). We recall that X, where $X \in \{F, B, L, R, U, D\}$ denotes a single turn of the cube. A sequence of turns consists of a combination of a certain number of single turns.

Let C be the set of all sequences of turns. Here we assume that two elements c_1, c_2 in C are equal if applied to the cube in P_0 they both result in the same configuration of the cube.

Note: Here we assume that two configurations are different, even if they can be turned into each other by rotating the cube in space.

1.) Write down how C can be given the structure of a group $(C, \cdot)$ and prove that it is indeed a group.

2.) Estimate the number of elements in $(C, \cdot)$.

3.) Is C an abelian group? If so, prove it, if not, give a counterexample.

4.) Let c be an element of the group C. Show that there is a minimal $n \in \mathbb{N} \setminus \{0\}$ such that $c^n = e$ and that the subgroup $\langle c \rangle$ generated by c has exactly n elements. Then find a non-trivial upper bound for the order $ord(c)$ of c.
Hint: Use the decomposition into disjoint cycles.

5.) **Principle of conjugation**: Let $(S_n, \circ)$ be the symmetric group on n elements and $\pi \in S_n$.
a) Let $p \in S_n$ be an element of $(S_n, \circ)$ and $a \in \{1, 2, ..., n\}$. Show the equivalence:

$$p(a) = a \Leftrightarrow \pi \circ p \circ \pi^{-1}(\pi(a)) = \pi(a).$$

b) Let $c = (a_1, a_2, ..., a_s) \in S_n$ be a cycle. Then $\pi \circ c \circ \pi^{-1}$ is the cycle $(\pi(a_1), \pi(a_2), ..., \pi(a_s))$.
c) $(C, \cdot)$ is isomorphic to a subgroup of a symmetric group. Which symmetric group(s)? Use this fact and part a) and b) to show that $FDF^{-1} \in C$ is a cycle and fixes four cubicles, i.e. four of the small cubes of the cube.
Note: You can use the principle of conjugation to find useful turns to solve the cube.

6.) Draw the Cayleygraph of the subgroup $H = \langle F^2, U^2 \rangle < C$ that is generated by the elements F^2 and U^2 and state the group to which H is isomorphic.

Observations

Write down some observations you made while working with the cube.
