Automorphism groups of graphs

Examples Find the automorphism groups of the following graphs.

We now prove that G is (isomorphic to) a subgroup of the automorphism group $\text{Aut}(\Gamma(G,S))$ of its Cayley graph $\Gamma(G,S)$. The proof is very similar to the proof of **Cayley's theorem**.

Theorem 10 Let $(G,\cdot)$ be a group and $S \subset G$, $\#S = n, n \in \mathbb{N}$ be a finite generating set, i.e. $\langle S \rangle = G$ and $\Gamma = \Gamma(G,S)$ be the corresponding Cayley graph. Then G is isomorphic to a subgroup G^* of $\text{Aut}(\Gamma(G,S))$:

$$G \simeq G^* \quad \text{where} \quad G^* < \text{Aut}(\Gamma(G,S)).$$

proof We have to find an injective group homomorphism $F : (G,\cdot) \to (\text{Aut}(\Gamma), \circ)$. This implies that $G \simeq F(G) = G^*$. We start with the construction of a map $F : G \to \text{Aut}(\Gamma)$ and then prove that F is a group homomorphism.

Step 1 For each $a \in G$ we construct a map $\rho_a = F(a) \in \text{Aut}(\Gamma)$:

We recall that for a Cayley graph Γ we have that $V(\Gamma) = G$ and $E(\Gamma) = G \times S$. For $a \in G$ set $\rho_a = (\rho_{a,V}, \rho_{a,E})$ where

a) $\rho_{a,V} : G \to G, a \mapsto \rho_{a,V}(x) = a \cdot x$.

b) $\rho_{a,E} : G \times S \to G \times S, a \mapsto \rho_{a,E}(x,s) = (a \cdot x, s)$.

To show that ρ_a is indeed an automorphism it is sufficient to show that that ρ_a is a morphism and that $\rho_{a,V}$ and $\rho_{a,E}$ are both bijective (see **Theorem 8**). We start with the latter condition:

- For fixed $a \in G$, the map $\rho_{a,V}(x) = a \cdot x$ is the multiplication from the left, which is bijective.
- For fixed $a \in G$, the map $\rho_{a,E} = \rho_{a,V} \times id$ is bijective as both $\rho_{a,V}$ and id are bijective (by **Lemma 6**).

- ρ_a satisfies the second condition for a morphism: for all $(x, s) \in G \times S$ we have

$$\delta(\rho_{a,E}(x, s)) = \rho_{a,V} \times \rho_{a,V}(\delta(x, s)).$$

proof By the definition of $\rho_{a,E}$ and the definition of the Cayley graph we have

$$\delta(\rho_{a,E}(x,s)) \overset{\textbf{Def.}\rho_{a,E}}{=} \delta(ax,s) \overset{\delta(g,s)=(g,gs)}{=} (ax, axs) \quad \text{and}$$
$$\rho_{a,V} \times \rho_{a,V}(\delta(x,s)) \overset{\delta(g,s)=(g,gs)}{=} \rho_{a,V} \times \rho_{a,V}(x, xs) = (\rho_{a,V}(x), \rho_{a,V}(xs)) = (ax, axs).$$

In total we have that $\rho_a \in \mathrm{Aut}(\Gamma)$ and the map F defined by $F(a) = \rho_a$ maps G into $\mathrm{Aut}(\Gamma)$.

Step 2 The map $F : (G, \cdot) \to (\mathrm{Aut}(\Gamma), \circ)$ is a group homomorphism

To show that F is a homomorphism we have to show that for all $a, b \in G$ we have

$$\rho_{ab} = F(a \cdot b) = F(a) \circ F(b) = \rho_a \circ \rho_b.$$

We know that $\rho_a = (\rho_{a,V}, \rho_{a,E})$, so we have to show that

$$\rho_{ab,V} = \rho_{a,V} \circ \rho_{b,V} \quad \text{and} \quad \rho_{ab,E} = \rho_{a,E} \circ \rho_{b,E}$$

proof For all $x \in G$ we have

$$\rho_{ab,V}(x) = abx \;\; \text{and} \;\; \rho_{a,V} \circ \rho_{b,V}(x) = \rho_{a,V}(bx) = abx$$

Hence $\rho_{ab,V}(x) = \rho_{a,V} \circ \rho_{b,V}(x)$ for all $x \in G$ and therefore $\rho_{ab,V} = \rho_{a,V} \circ \rho_{b,V}$.
To prove the statement for $\rho_{ab,E}$ it is sufficient to see that $\rho_{ab,E} = \rho_{ab,V} \times id$ and the proof folllows from the first part. In total this implies that F is a group homomorphism.

Step 3 The map $F : (G, \cdot) \to (\mathrm{Aut}(\Gamma), \circ)$ is injective

proof To show that F is injective we have to show that for all $a, b \in G$ we have

$$\rho_a = F(a) = F(b) = \rho_b \Rightarrow a = b.$$

But if $\rho_a = \rho_b$ then $\rho_{a,V} = \rho_{b,V}$. Especially for $x = e$, where e is the neutral element of G we have

$$a = ae = \rho_{a,V}(e) = \rho_{b,V}(e) = be = b.$$

Hence $a = b$. This implies that F is injective.

In total F is an injective group homomorphism and therefore $F : G \to F(G)$ is a bijective group homomorphism. This means that

$$G \simeq F(G) = G^* < \text{Aut}(\Gamma(G, S))$$

and we have shown that G is (isomorphic to) a subgroup G^* of the automorphism group $\text{Aut}(\Gamma(G, S))$ of its Cayley graph. □

Examples: For the following Cayleygraphs we have:

i) For $\Gamma_1 = \Gamma(\mathbb{Z}_6, \{1\})$ we have that $\text{Aut}(\Gamma_1) = \mathbb{Z}_6$. Can you describe the effect of $\rho_a : \Gamma_1 \to \Gamma_1$ on the graph for an $a \in (\mathbb{Z}_6, +_6)$?

ii) For $\Gamma_2 = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1, 0), (0, 1)\})$ we have that $\mathbb{Z}_2 \times \mathbb{Z}_2 < \text{Aut}(\Gamma_1)$ and $\text{Aut}(\Gamma_1) \simeq D_8$. Can you describe the effect of $\rho_a : \Gamma_2 \to \Gamma_2$ on the graph for an $a \in (\mathbb{Z}_2 \times \mathbb{Z}_2, +_2 \times +_2)$?

Note 11 1.) For all $a \in G \backslash \{e\}$ we have that $\rho_a : \Gamma \to \Gamma$ does not fix any vertex, i.e.

$$\rho_{a,V}(x) \neq x \quad \text{for all} \quad x \in G.$$

as if $\rho_{a,V}(x) = x$ then $\rho_{a,V}(x) = ax = x$. But $ax = x$ implies that $a = e$. Therefore only $\rho_e = id$ fixes vertices.

2.) For all $a \in G$ we have that $\rho_a(e) = ae = a$. This means that for any vertex $a \in G$ there is a symmetry or automorphism that sends e to a.

This means that a Cayley graph is a **homogeneous space**: It "looks the same" from any vertex.

Outlook Conversely many homogeneous graphs and spaces can be seen as groups. Examples are the circle, the line or the plane.