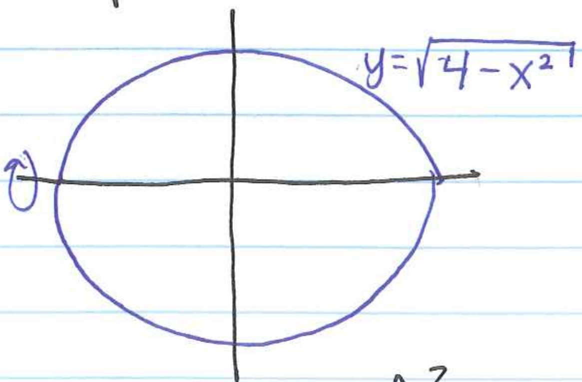
Example 1 Surface area of a sphere with radius 2.



$$\int_{-2}^{2} 2\pi\sqrt{4-x^2}\sqrt{1+\left(\frac{-2x}{\cancel{2}\sqrt{4-x^2}}\right)^2}\,dx$$

$$y' = f'(x) = \frac{-2x}{2\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} 2\pi\sqrt{(4-x^2)\left(1+\frac{x^2}{4-x^2}\right)}\,dx$$

$$= \int_{-2}^{2} 2\pi\sqrt{4-x^2+x^2}\,dx = \int_{-2}^{2} 2\pi\sqrt{4}\,dx$$

$$= \int_{-2}^{2} 4\pi\,dx$$

$$= 4\pi x\Big|_{-2}^{2} = \boxed{16\pi}$$

Rotating around different axes and integrating with respect to different variable.

* You can integrate with respect to x <u>or</u> y when rotating around x-axis or y-axis *

rotating about	$y=f(x)$	$x=g(y)$
x-axis	$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$	$\int_c^d 2\pi y\sqrt{1+(g'(y))^2}\,dy$
y-axis	$\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx$	$\int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$

Solution 2: In terms of y: $y=x^2 \Rightarrow x=\sqrt{y}$

$$\int_1^4 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy = \int_1^4 2\pi\sqrt{y\left(1+\frac{1}{4y}\right)}\,dy$$

$$= \int_1^4 2\pi\sqrt{y+\frac{1}{4}}\,dy$$

$$= 2\pi\left(\frac{2}{3}\right)\left(y+\frac{1}{4}\right)^{3/2}\Big|_1^4$$

$$= \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2}\right]$$

$$= \frac{4\pi}{24}\left(17\sqrt{17}-5\sqrt{5}\right) = \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

Example: Find the area of the surface obtained by rotating $y=x^3$ about the x-axis between $0 \le x \le 2$.

In terms of x: $$\int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1+9x^4}\,dx$$

$u = 1+9x^4 \quad du = 36x^3\,dx$

$$= \int_{x=1}^{x=2} \frac{\pi}{18}\sqrt{u}\,du = \frac{\pi}{18}\left(\frac{2}{3}\right)u^{3/2}\Big|_{x=1}^{x=2}$$

$$= \frac{\pi}{27}(1+9x^4)^{3/2}\Big|_1^2$$

$$= \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

Example | Find the S.A. of $y = \frac{x^3}{3} + \frac{1}{4x}$ $1 \le x \le 2$ rotated about y-axis.

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

$$S.A. = \int_1^2 2\pi x \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \sqrt{1 + x^4 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16x^4}}\, dx$$

$$= \int_1^2 2\pi x \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx$$

$$= \int_1^2 2\pi x \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \left(x^2 + \frac{1}{4x^2}\right) dx$$

$$= \int_1^2 2\pi \left(x^3 + \frac{1}{4x}\right) dx$$

$$= 2\pi \left(\frac{x^4}{4} + \frac{1}{4}\ln x\right) \Big|_1^2$$

$$= 2\pi \left(4 + \frac{\ln(2)}{4} - \frac{1}{4}\right)$$

- Find the area of the surface obtained by rotating the curve $x = 1+2y^2$, $1 \le y \le 2$ about the x-axis.

$$\int_1^2 2\pi y\sqrt{1+(4y)^2}\,dy = \int_1^2 2\pi y\sqrt{1+16y^2}\,dy \qquad \begin{matrix} u = 1+16y^2 \\ du = 32y\,dy \end{matrix}$$

$$= \int_{y=1}^{y=2} \frac{\pi}{16}\sqrt{u}\,du$$

$$= \frac{\pi}{16}\left(\frac{2}{3}\right)u^{3/2}\Big|_{y=1}^{y=2} = \frac{\pi}{24}(1+16y^2)^{3/2}\Big|_1^2$$

$$= \frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$$

- Find the same area as above, but now with the integral in terms of x. (or y, if you did the above with x).

$x = 1+2y^2 \Longleftrightarrow \sqrt{\frac{x-1}{2}} = y \qquad 1 \le y \le 2 \Longleftrightarrow 3 \le x \le 9$

$$\int_3^9 2\pi\sqrt{\frac{x-1}{2}}\sqrt{1+\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2\sqrt{x-1}}\right)^2}\,dx$$

$$= \int_3^9 2\pi\sqrt{\left(\frac{x-1}{2}\right)\left(1+\frac{1}{2}\cdot\frac{1}{4(x-1)}\right)}\,dx$$

$$= \int_3^9 2\pi\sqrt{\frac{x-1}{2}+\frac{1}{16}}\,dx \qquad u = \frac{x}{2}-\frac{7}{16} \quad du = \frac{1}{2}\,dx$$

$$= \int_3^9 2\pi\sqrt{\frac{x}{2}-\frac{7}{16}}\,dx = \int_3^9 2\pi\cdot 2\sqrt{u}\,du = 4\pi\left(\frac{2}{3}\right)u^{3/2}\Big|_3^9 = \frac{8\pi}{3}\left(\frac{x}{2}-\frac{7}{16}\right)^{3/2}\Big|_3^9$$

$$= \frac{8}{3}\pi\left(\frac{65\sqrt{65}}{64} - \frac{17\sqrt{17}}{64}\right)$$