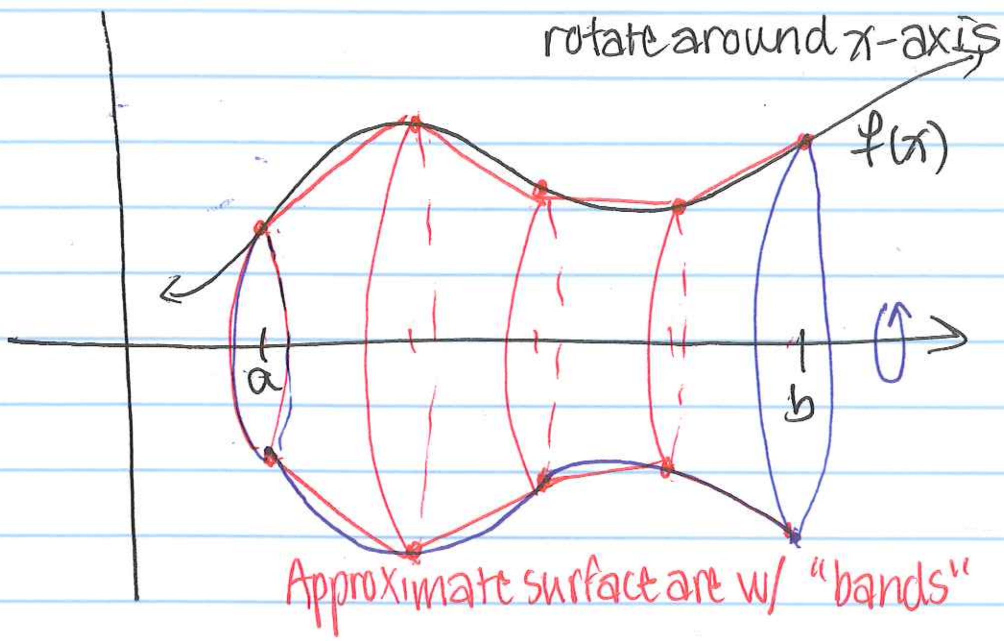
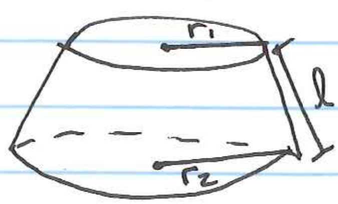
March 4, 2013

Announcements:

Last Time: Surface Area of a Solid of Revolution.



Surface area of a Band:



$A = 2\pi r l$

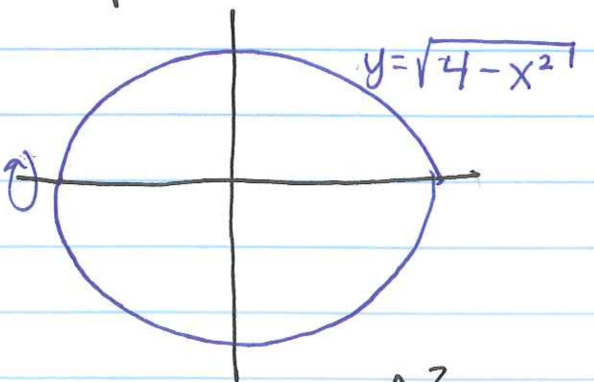
where $r = \dfrac{r_1 + r_2}{2}$

As our estimate gets better and we use more bands, r_1, r_2 will be closer together.

$2\pi r l \approx 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$

Surface Area: $\int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx$

Example 1 Surface area of a sphere with radius 2.



$$\int_{-2}^{2} 2\pi\sqrt{4-x^2}\sqrt{1+\left(\frac{-2x}{\cancel{2}\sqrt{4-x^2}}\right)^2}\,dx$$

$$y' = f'(x) = \frac{-2x}{2\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} 2\pi\sqrt{(4-x^2)\left(1+\frac{x^2}{4-x^2}\right)}\,dx$$

$$= \int_{-2}^{2} 2\pi\sqrt{4-x^2+x^2}\,dx = \int_{-2}^{2} 2\pi\sqrt{4}\,dx$$

$$= \int_{-2}^{2} 4\pi\,dx$$

$$= 4\pi x \Big|_{-2}^{2} = \boxed{16\pi}$$

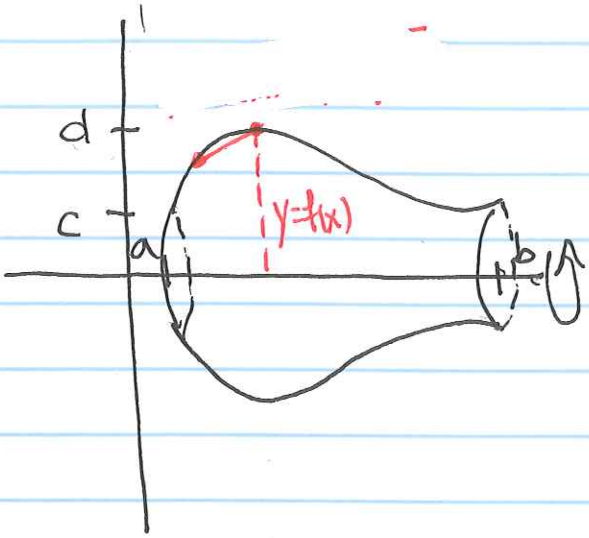
Rotating around different axes and integrating with respect to different variable.

* You can integrate with respect to x <u>or</u> y when rotating around x-axis or y-axis *

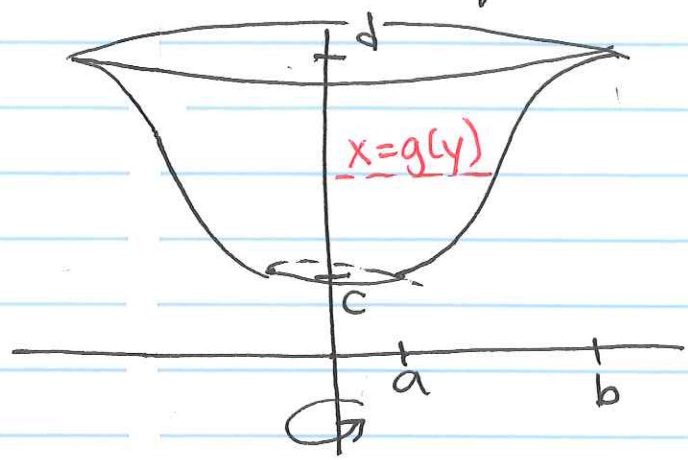
rotating about	$y=f(x)$	$x=g(y)$
x-axis	$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$	$\int_c^d 2\pi y\sqrt{1+(g'(y))^2}\,dy$
y-axis	$\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx$	$\int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$

The picture that goes with the chart:

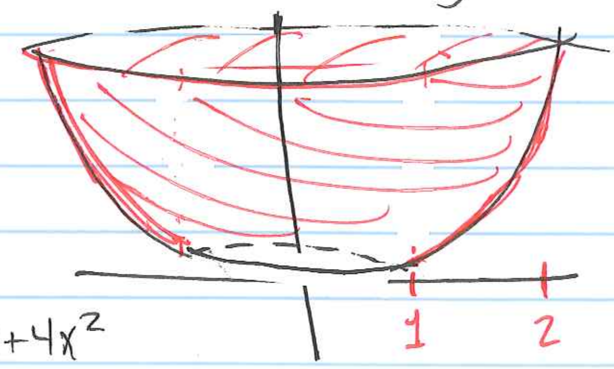
rotate around x-axis



rotate around y-axis



Example The arc of the parabola $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface.



Solution 1 Arc length

In terms of x

$$= \int_1^2 2\pi(x)\sqrt{1+(2x)^2}\,dx$$

$$= \int_1^2 2\pi x\sqrt{1+4x^2}\,dx \qquad u=1+4x^2$$

$$du = 8x\,dx$$

$$= \int_{x=1}^{x=2} \frac{\pi}{4}\sqrt{u}\,du$$

$$= \frac{\pi}{4}\left(\frac{2}{3}\right)u^{3/2}\Big|_{x=1}^{x=2} = \frac{\pi}{6}(1+4x^2)^{3/2}\Big|_1^2 = \frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$$

$$= \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

Solution 2: In terms of y: $y=x^2 \Rightarrow x=\sqrt{y}$

$$\int_1^4 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy = \int_1^4 2\pi\sqrt{y\left(1+\frac{1}{4y}\right)}\,dy$$

$$= \int_1^4 2\pi\sqrt{y+\frac{1}{4}}\,dy$$

$$= 2\pi\left(\frac{2}{3}\right)\left(y+\frac{1}{4}\right)^{3/2}\Big|_1^4$$

$$= \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2}-\left(\frac{5}{4}\right)^{3/2}\right]$$

$$= \frac{4\pi}{24}\left(17\sqrt{17}-5\sqrt{5}\right) = \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

Example: Find the area of the surface obtained by rotating $y=x^3$ about the x-axis between $0\le x\le 2$.

In terms of x: $$\int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1+9x^4}\,dx$$

$u = 1+9x^4 \quad du = 36x^3\,dx$

$$= \int_{x=1}^{x=2} \frac{\pi}{18}\sqrt{u}\,du = \frac{\pi}{18}\left(\frac{2}{3}\right)u^{3/2}\Big|_{x=1}^{x=2}$$

$$= \frac{\pi}{27}(1+9x^4)^{3/2}\Big|_1^2$$

$$= \frac{\pi}{27}\left(145^{3/2}-10^{3/2}\right)$$

$$= \frac{\pi}{27}\left(145\sqrt{145}-10\sqrt{10}\right)$$