Arc Length Function

Given a function $f(t)$ and starting point $(a, f(a))$.

the arc length function $s(x)$ is defined:

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$$

* this is just the "arc-length so far" function *

Example Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $(1,1)$ as the starting point.

$$\frac{dy}{dx} = 2x - \frac{1}{8x}$$

Arc length function: $s(x) = \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2}\, dt$

$$= \int_1^x \sqrt{1 + \left(4t^2 - \frac{1}{2} + \frac{1}{64t^2}\right)}\, dt$$

$$= \int_1^x \sqrt{4t^2 + \frac{1}{2} + \frac{1}{64t^2}}\, dt$$

this factors:

$$4t^2 + \frac{1}{2} + \frac{1}{64t^2} = \left(2t + \frac{1}{8t}\right)^2$$

$$= \int_1^x 2t + \frac{1}{8t}\, dt$$

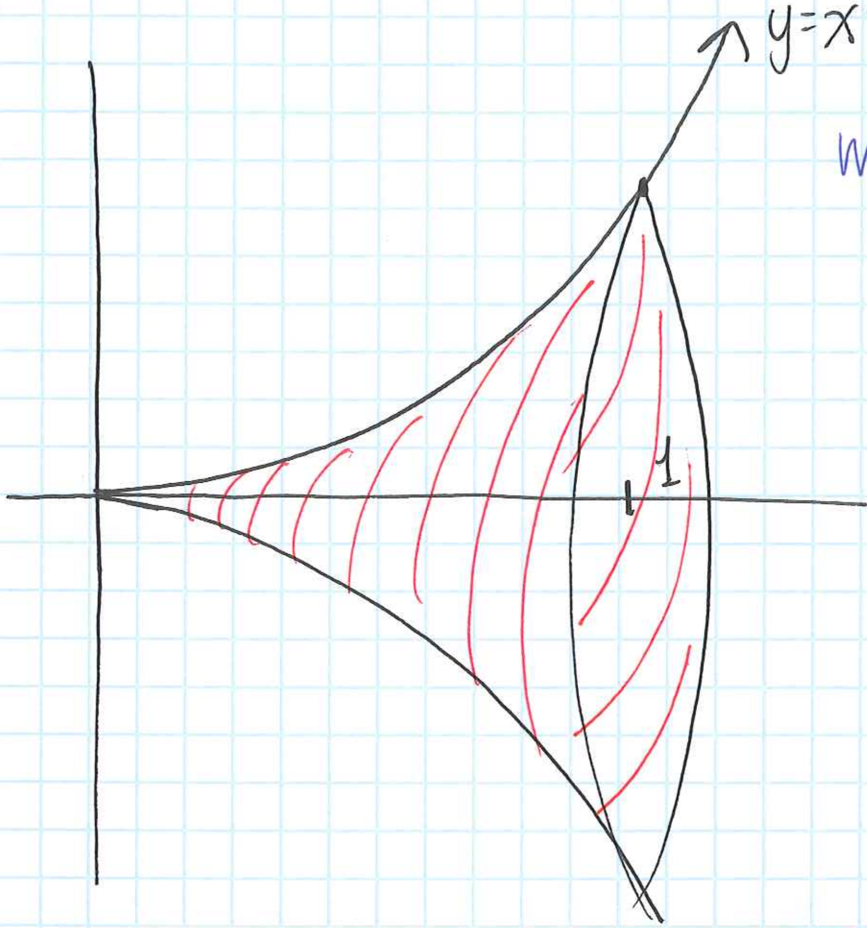
$$= t^2 + \frac{1}{8}\ln t \Big|_1^x = x^2 + \frac{1}{8}\ln x - 1$$

Now we could use our arc length function.

Arc Length from $x=1$ to $x=5$: $s(5)=25+\frac{1}{8}\ln 5-1$

" $x=1$ to $x=7$: $s(7)=49+\frac{1}{8}\ln 7-1$

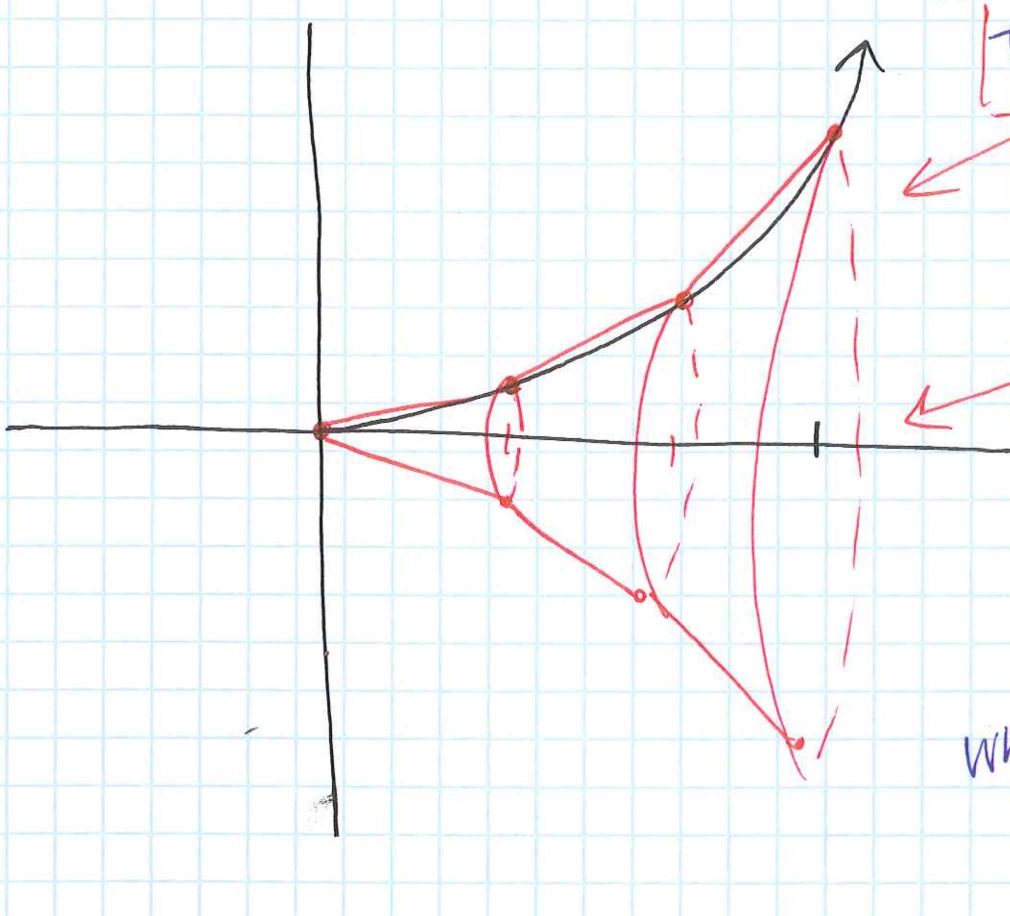
Area of a Surface of Revolution



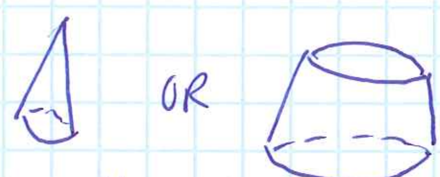
want the area of the surface that is created.

Think about how we got arc length

Now rotate these estimating lines about the x-axis.

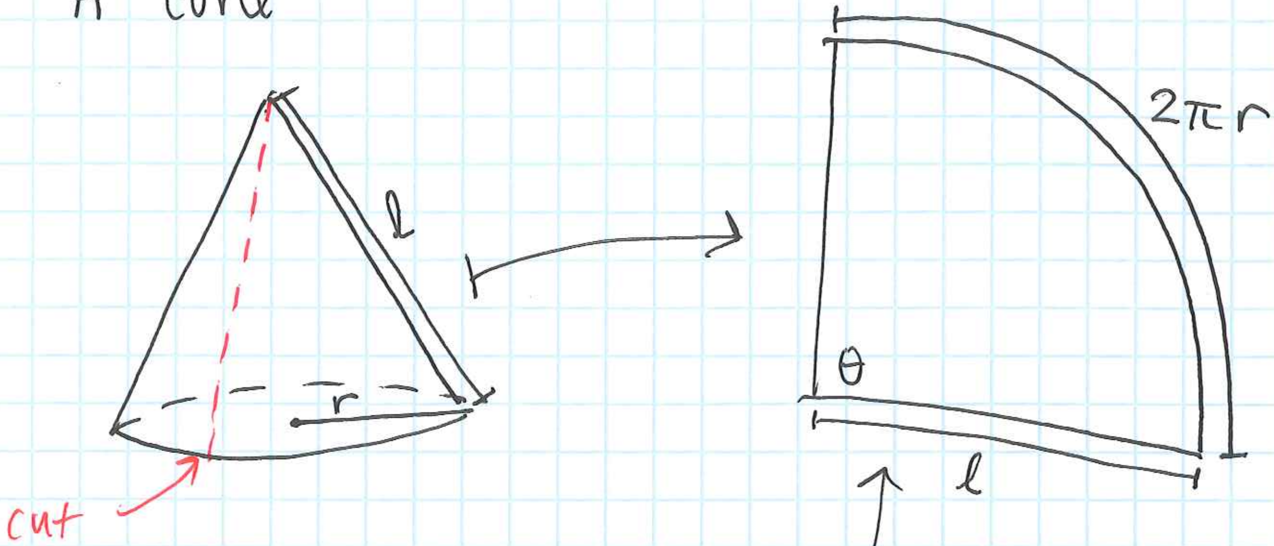


We get shapes that look like



What is the surface area of these shapes?

Surface Area of A cone



We need the area of this sector.

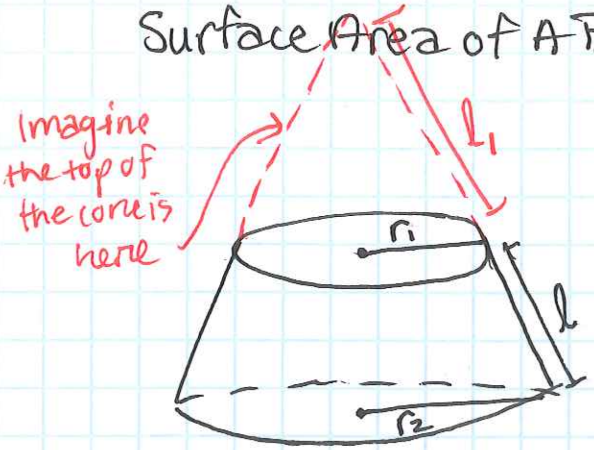
$$\text{Area of sector} = \frac{\theta}{2\pi} \cdot \pi l^2$$

ratio of sector to entire circle.

$$\frac{\theta}{2\pi} = \frac{\text{arc length of sector}}{\text{circumference of entire circle}} = \frac{2\pi r}{2\pi l} = \frac{r}{l}$$

$$\text{so Area of sector} = \frac{r}{l} \cdot \pi l^2 = \underline{\underline{\pi r l}}$$

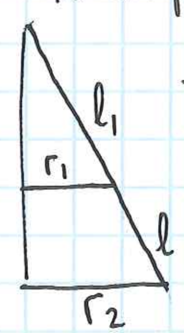
Surface Area of A Band



Surface Area = SA of big cone − SA of little cone

$$= \pi r_2 (l + l_1) - \pi r_1 l_1$$

to simplify, use similar triangles.



$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2} \iff r_2 l_1 = r_1 (l_1 + l)$$

So ... $\pi r_2(l+l_1) - \pi r_1 l_1 = \pi r_2 l + \underbrace{\pi r_2 l_1}_{\pi(r_1(l_1+l))\text{ by similar triangles}} - \pi r_1 l_1$

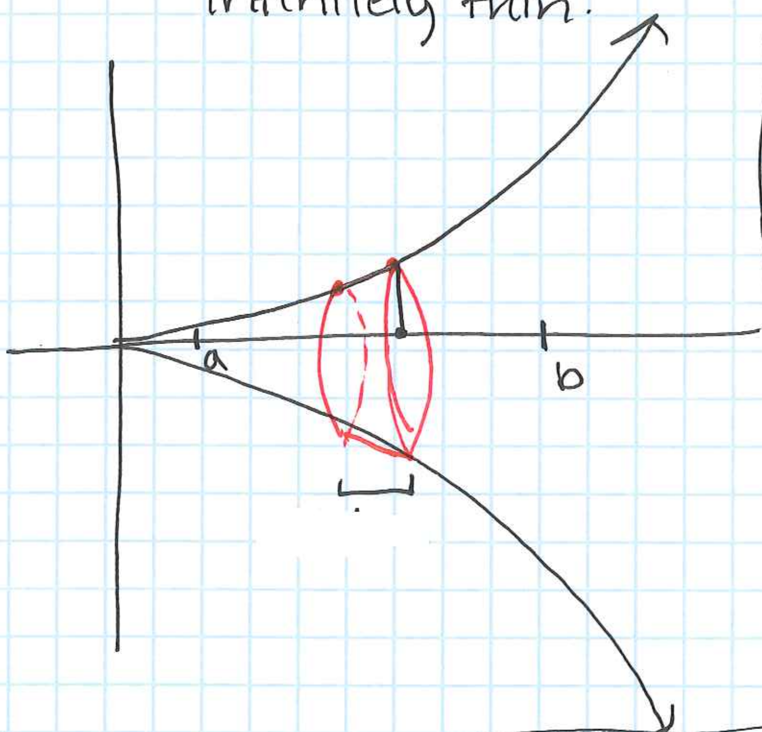
$$= \pi r_2 l + \pi r_1 l_1 + \pi r_1 l - \pi r_1 l_1$$

$$= \pi l(r_1+r_2)$$

$$= 2\pi r l \quad \text{where } r = \frac{r_1+r_2}{2}$$

Now let's go back to our estimation of surface area.

- We want to take an "infinite" # of bands to estimate surface area.
- As this happens, their width becomes infinitely thin.



$\boxed{2\pi r l}$

l = length of line
$= \sqrt{1+(f'(x))^2}\,dx$

r = height of graph
$= f(x)$

$$\boxed{\text{Surface Area} = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx}$$