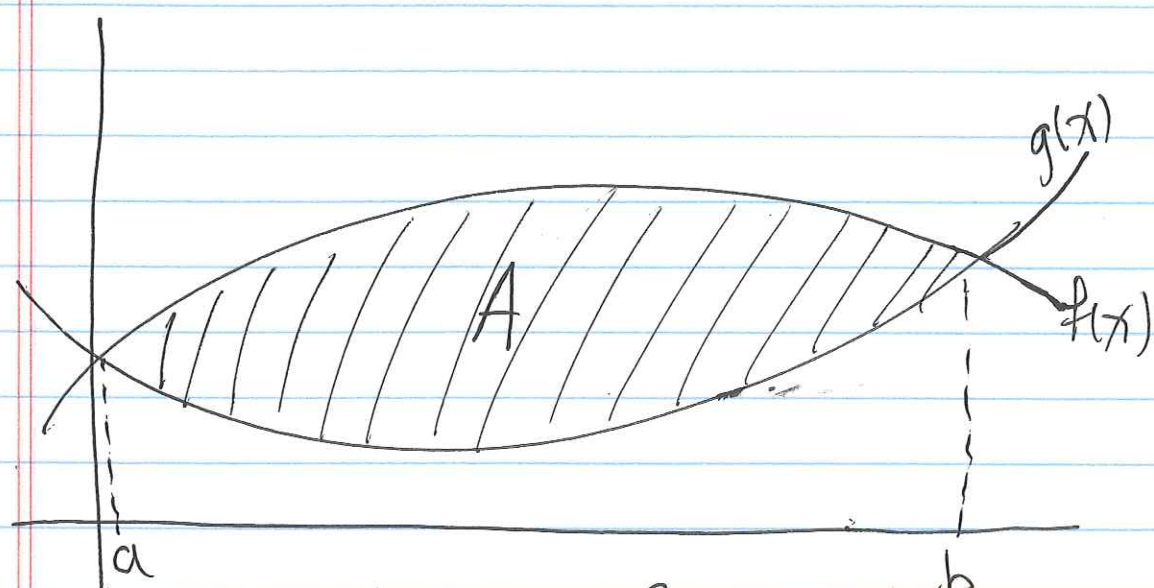
Announcements

- Midterm: Integral table will be provided
  Know your basic trig identities
- HW2 Due today
- 5.5 Substitution Rule Due Monday (1/28)

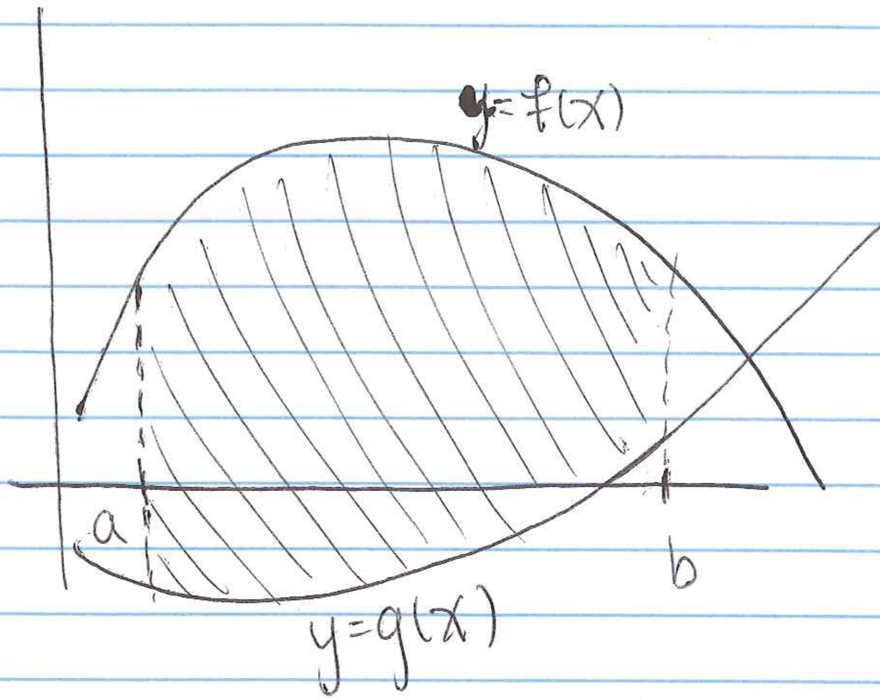


What is the area of A?

$$\int_a^b (f(x) - g(x))\,dx$$

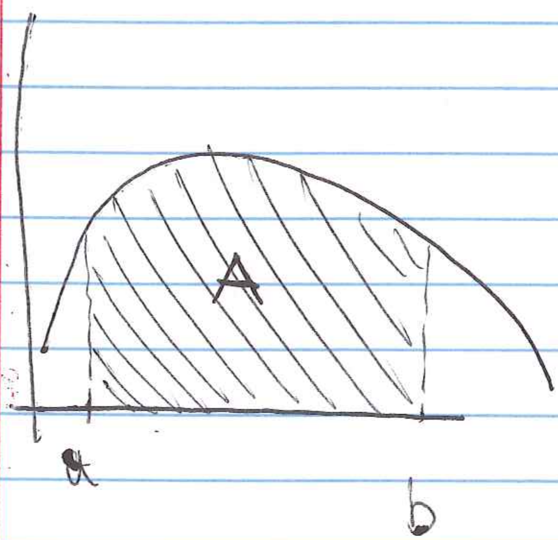
Area Between Curves

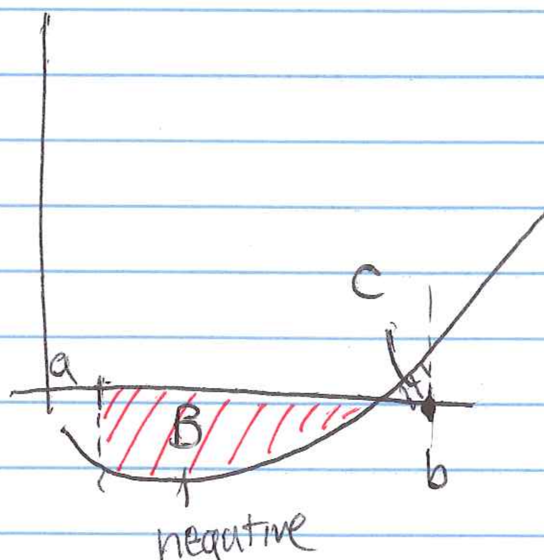
We want the area as follows:



$\int_a^b f(x)\,dx$:

$\int_a^b g(x)\,dx$:





The area we want is

$$\underbrace{A+}_{\int_a^b f(x)dx}\ \underbrace{B-C}_{-\int_a^b g(x)dx} \Rightarrow \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

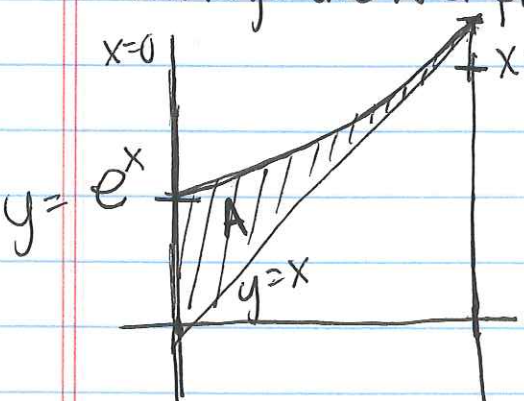
DEF: The area A of a region bounded by the curves $y=f(x)$, $y=g(x)$, and the lines $x=a$, $x=b$ where f, g are continuous and $f(x) \geq g(x)$ for all x in $[a,b]$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Examples:

(1) Find the area of the region bounded above by $y=e^x$, bounded below by $y=x$, and bounded on the sides by $x=0$ and $x=1$.

Always draw a picture



$$A = \int_0^1 (e^x - x)\,dx$$

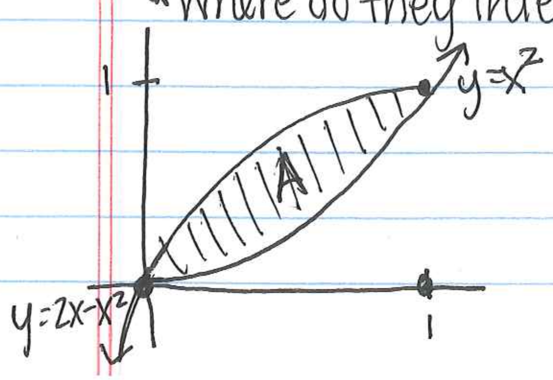
$$= e^x - \frac{x^2}{2}\Big|_0^1$$

$$= e^1 - \frac{1^2}{2} - e^0 + \frac{0}{2}$$

$$= e - \frac{1}{2} - 1 = \boxed{e - \frac{3}{2}}$$

(2) (Enclosed Area) Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$

*where do they intersect?



-set them equal:

$$x^2 = 2x - x^2$$

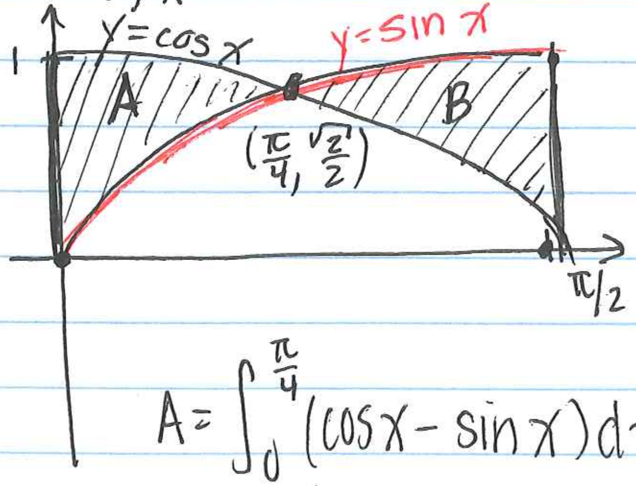
$$2x^2 - 2x = 0$$

$$x(2x-2) = 0$$

$$x = 0 \text{ OR } x = 1$$

$$A = \int_0^1 ((2x - x^2) - x^2)\,dx = x^2 - \frac{2}{3}x^3 \Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

(3) Find the area of the region bounded by $y = \sin x$, $y = \cos x$
$x = 0$, $x = \pi/2$



Need to take two different integrals:
what value is the intersection pt?
$\sin x = \cos x$
when $x = \pi/4$

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx = \sin x + \cos x \Big|_0^{\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1$$

$$B = \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx = -\cos x - \sin x \Big|_{\pi/4}^{\pi/2}$$
$$= 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$A + B = \sqrt{2} - 1 + (\sqrt{2} - 1)$$
$$= \boxed{2\sqrt{2} - 2}$$

The area between the curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\,dx$$

<u>The process:</u>

(1) decide on what intervals $f(x)$ or $g(x)$ is the largest
(2) Integrate $f(x) - g(x)$ over the intervals $f(x)$ is larger
$g(x) - f(x)$ over the intervals $g(x)$ is larger
(3) Add the integrals together

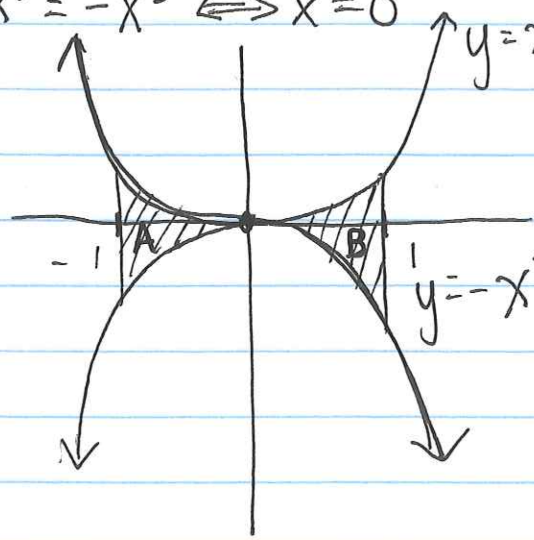
Things you should do when calculating area between curves

(in whatever order works best for you)

- Find where curves intersect (set them equal and solve)
- Using intersection points, figure out ~~when~~ on what intervals each curve is "on top"
- Sketch a graph, determine the area you're finding
- Integrate appropriately; if you get a negative #, something went wrong

Example (Practice): (4) $y=x^3,\ y=-x^3,\ x=-1,\ x=1$

$x^3=-x^3 \iff x=0$

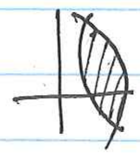


$$\int_{-1}^{0}(-x^3-x^3)\,dx+\int_{0}^{1}(x^3+x^3)\,dx$$

$$=-\frac{2x^4}{4}\Big|_{-1}^{0}+\frac{2x^4}{4}\Big|_{0}^{1}$$

$$=\frac{2(-1)^4}{4}+\frac{2(1)^4}{4}=\boxed{1}$$

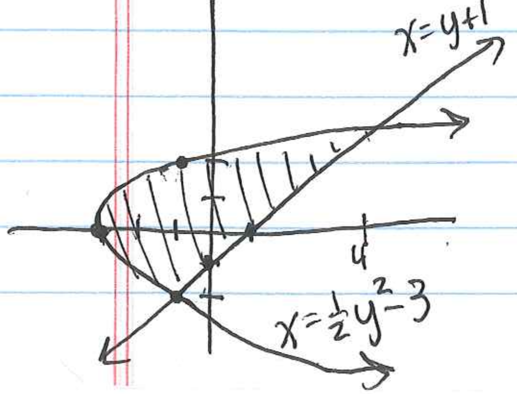
Sometimes we get "vertical" area"



then you want to make x a function of y.

(5) Find the area enclosed by the line $y=x-1$ and the parabola $y^2=2x+6$.

~~points of intersection~~ Want to solve for x, so we think of the area the rotated way.



$y^2=2x+6 \iff \frac{1}{2}y^2-3=x$

$y=x-1 \iff y+1=x$

Think of rotating your page so the positive x-axis points up

So $x=y+1$ is "above" $x=\frac{1}{2}y^2-3$

Intersection points:

$$y+1=\frac{1}{2}y^2-3$$

$$0=\frac{1}{2}y^2-y-4 \Longleftrightarrow 0=y^2-2y-8 \Longleftrightarrow 0=(y-4)(y+2)$$

$$y=4 \text{ OR } y=-2$$

$$\int_{-2}^{4}(y+1)-\left(\tfrac{1}{2}y^2-3\right)dy=\frac{y^2}{2}+4y-\frac{y^3}{6}\Big|_{-2}^{4}$$

$$=\frac{4^2}{2}+4\cdot4-\frac{4^3}{6}-\frac{(-2)^2}{2}-4(-2)+\frac{(-2)^3}{6}$$

$$=8+16-\frac{64}{6}-2+8-\frac{8}{6}=18$$

(6) $y=\sqrt{x-1}$ $\quad x-y=1$

$y=x-1$

$$\sqrt{x-1}=x-1$$

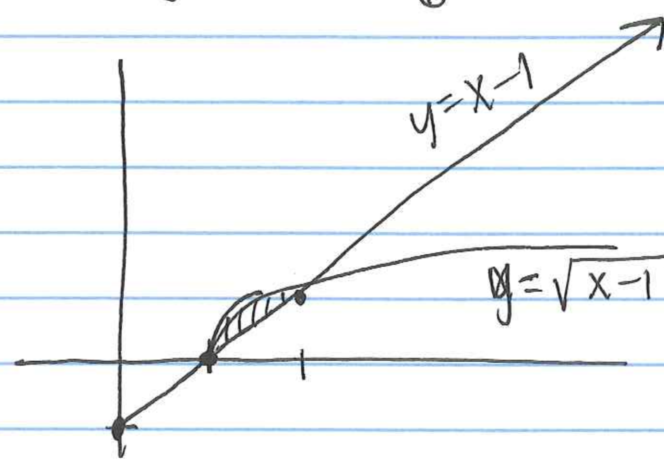
$$x-1=(x-1)^2$$

$$x-1=x^2-2x+1$$

$$0=x^2-3x+2$$

$$=(x-1)(x-2)$$

$$x=1,\ x=2$$



$$\int_{1}^{2}\left(\sqrt{x-1}-(x-1)\right)dx=\frac{2}{3}(x-1)^{3/2}-\frac{x^2}{2}+x\Big|_{1}^{2}$$

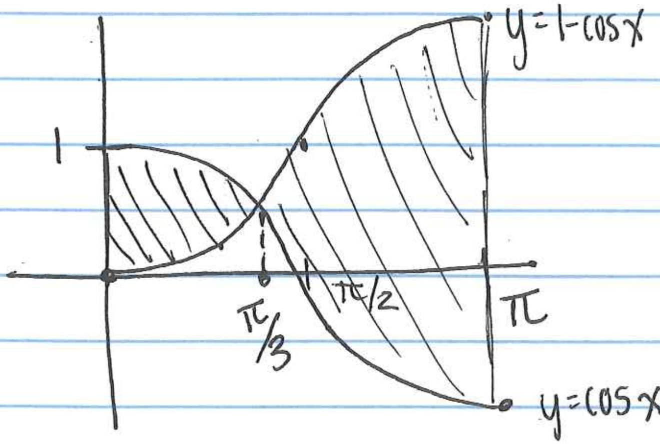
$$=\frac{2}{3}1^{3/2}-\frac{2^2}{2}+2-0+0+0$$

$$=\frac{2}{3}-2+2=\boxed{\frac{2}{3}}$$

(7) $y=\cos x$, $y=1-\cos x$, $0 \le x \le \pi$

Intersection: $\cos x = 1-\cos x \iff \cos x = \frac{1}{2}$

$x = \pi/3$



$$\int_0^{\pi/3} \cos x-(1-\cos x)\,dx + \int_{\pi/3}^{\pi} (1-\cos x)-\cos x \, dx$$

$$= 2\sin x - x \Big|_0^{\pi/3} + x-2\sin x \Big|_{\pi/3}^{\pi}$$

$$= 2\cdot\frac{\sqrt{3}}{2} - \frac{\pi}{3} - 0 + 0 + \pi - 0 - \frac{\pi}{3} + 2\cdot\frac{\sqrt{3}}{2}$$

$$= 2\sqrt{3} - \frac{2\pi}{3} + \pi = 2\sqrt{3} + \frac{\pi}{3}$$

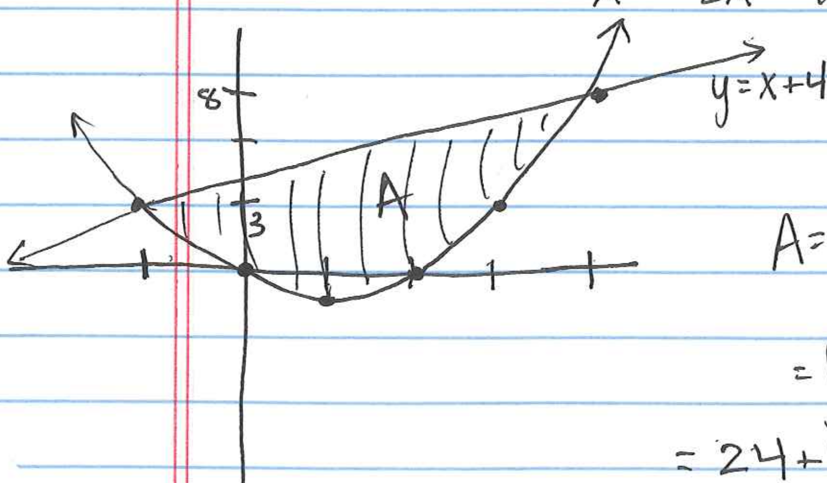
(Practice) (8) ~~$y = e$~~

$y=(x^2-2x)$, $y=x+4$

Intersection: $x^2-2x=x+4$

$x^2-3x-4=0 \iff (x-4)(x+1)$

$x=4$ OR $x=-1$



$$A=\int_{-1}^{4} (x+4)-(x^2-2x)\,dx$$

$$=\int_{-1}^{4} 3x+4-x^2\,dx = \frac{3}{2}x^2+4x-\frac{x^3}{3}\Big|_{-1}^{4}$$

$$= 24+16-\frac{64}{3}-\frac{3}{2}+4-\frac{1}{3}$$