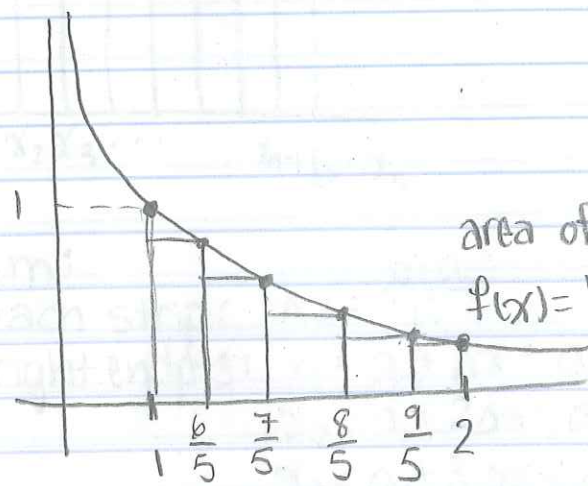
Fri, Jan. 11, 2013

Recall: Area under a curve
- Right endpts
- Left endpoints
- Midpts (useful for approximating)

- How do we make our estimates better?
  Use more rectangles
  Applet: www.slu.edu/classes/maymk/Riemann.html

The (Precise) Definition of Area

Example:



area of $f(x) = 1/x$ from $x=1$ to $x=2$ using 5 rect.

Let's use right end points:

- width of a strip: $\Delta x = \frac{2-1}{5} = \frac{1}{5}$

- x-values of right endpoints: $1+\frac{1}{5}, 1+\frac{2}{5}, 1+\frac{3}{5}, 1+\frac{4}{5}, 1+\frac{5}{5}$

  $\quad = \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, \frac{10}{5} = 2$

- $R_5 = f(\frac{6}{5})\Delta x + f(\frac{7}{5})\Delta x + f(\frac{8}{5})\Delta x + f(\frac{9}{5})\Delta x + f(2)\Delta x$

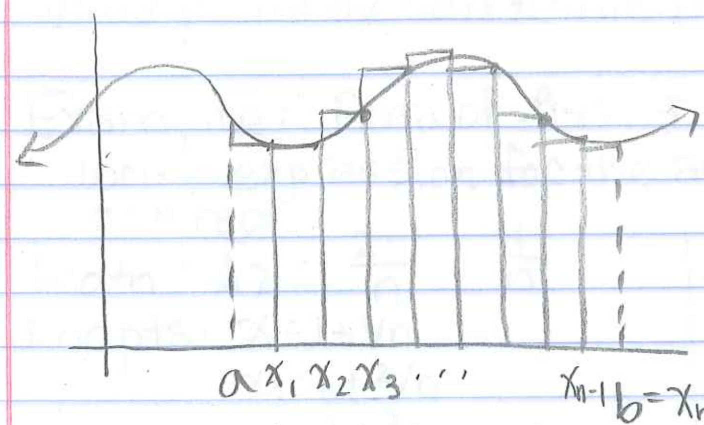
$$= \frac{1}{(6/5)} \cdot \frac{1}{5} + \frac{1}{(7/5)} \cdot \frac{1}{5} + \frac{1}{(8/5)} \cdot \frac{1}{5} + \frac{1}{(9/5)} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{5}$$

$$= \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} = .645635$$

Goal: Define area w/ "infinitely" many rectangles

We have: Curve $f(x)$
Bounds a, b $(a \le b)$
Rectangles n

Want: Area under $f(x)$ from $x=a$ to $x=b$



Riemann Sum:
Width of each strip: $\Delta x = \frac{b-a}{n}$
x-values of right endpts: $x_1 = a + \Delta x = a + \frac{b-a}{n}$
$x_2 = a + 2\Delta x = a + 2\left(\frac{b-a}{n}\right)$
$x_3 = a + 3\Delta x = a + 3\left(\frac{b-a}{n}\right)$
$\vdots$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x$$

<u>DEF</u>: The <u>area</u> A of a region that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x\right]$$

Sometimes you'll see $x_1^*, x_2^*, x_3^*, \ldots$ instead of $x_1, x_2, x_3, \ldots$

$x_1^*, x_2^*, x_3^*, \ldots$ are sample points taken in each strip - random instead of systematic like right, left endpts or midpts.

* This implies the it ~~doesn't~~ matter what rule we use, when we take the limit we should always get the same thing *

Example: Area of $f(x) = \frac{1}{x}$ from $x=1$ to $x=2$

write expression for the area:

$n = \#$ rect.

Width: $\Delta x = \frac{2-1}{n} = \frac{1}{n}$

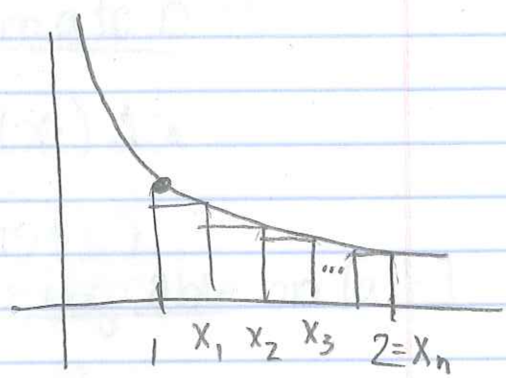
Endpts: $x_1 = 1 + 1/n$

$x_2 = 1 + 2/n$

$x_3 = 1 + 3/n$

$\vdots$

$x_n = 1 + n/n = 2$



$$A = \lim_{n\to\infty} \left[f\left(1+\tfrac{1}{n}\right)\tfrac{1}{n} + f\left(1+\tfrac{2}{n}\right)\tfrac{1}{n} + f\left(1+\tfrac{3}{n}\right)\tfrac{1}{n} + \ldots + f(2)\tfrac{1}{n} \right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{1}{(1+\frac{1}{n})}\right)\frac{1}{n} + \left(\frac{1}{(1+\frac{2}{n})}\right)\frac{1}{n} + \left(\frac{1}{(1+\frac{3}{n})}\right)\frac{1}{n} + \ldots + \left(\frac{1}{2}\right)\frac{1}{n} \right]$$

A litte extra notation:

Writing a sum more succinctly

$$\lim \left[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x \right]$$

$$\sum_{i=1}^{n} f(x_i)\Delta x =$$

tells us to add $\longrightarrow$ $\sum$

$n \leftarrow$ stop when $i = n$

$i=1$: Start by letting $i=1$, then 2, then 3, ...

So $A = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\,\Delta x\right]$

The Definite Integral

- f is a function defined for $a \le x \le b$
- divide interval $[a,b]$ into strips (subintervals) of equal width: $\Delta x = (b-a)/n$
- $x_0 (=a), x_1, x_2, \dots, x_n (=b)$ are the endpts of these subintervals

The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

(provided this limit exists)
If it does, we say f is integrable on $[a,b]$

*formal def uses x_i^* to indicate sample points where i is in $[x_{i-1}, x_i]$.
* we are using endpts because this is the same as sample pts whenever f is integrable

When can we use integrals?
- when f is continuous on $[a,b]$
- when f only has a fin. # of jump discontinuities

Anatomy of def int.

$$\int_{a}^{b} \underbrace{f(x)}_{\text{function}}\, \underbrace{dx}$$

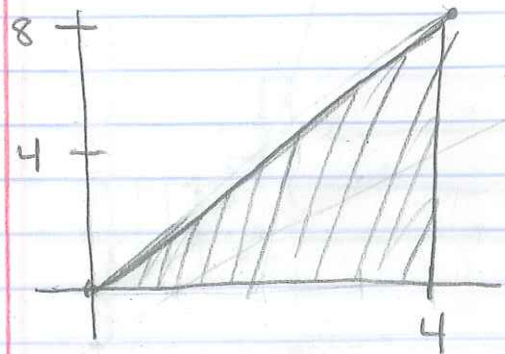
upper bound $\rightarrow b$

lower bound $\rightarrow a$

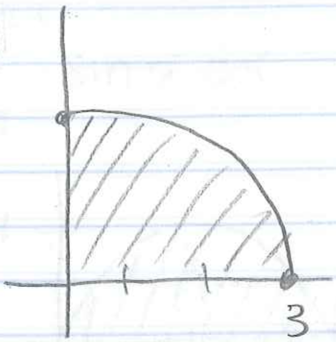
function

think of this as integral version of Δx. tells us what variable to integrate w/ respect to.

Examples: (1) $\int_0^4 (2x)\,dx = \frac{1}{2}(4 \cdot 8) = 16$



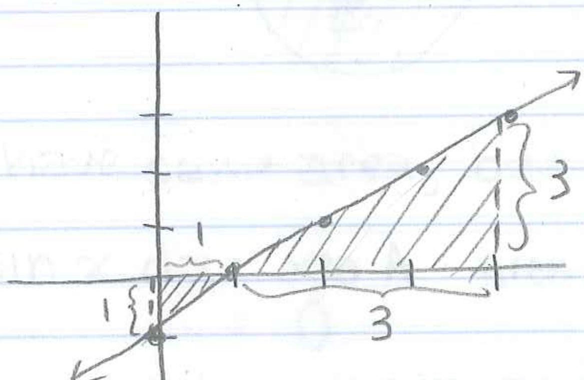
(2)



$$\int_0^3 \sqrt{9-x^2}\,dx$$

$$= \frac{9\pi}{4}$$

(3)



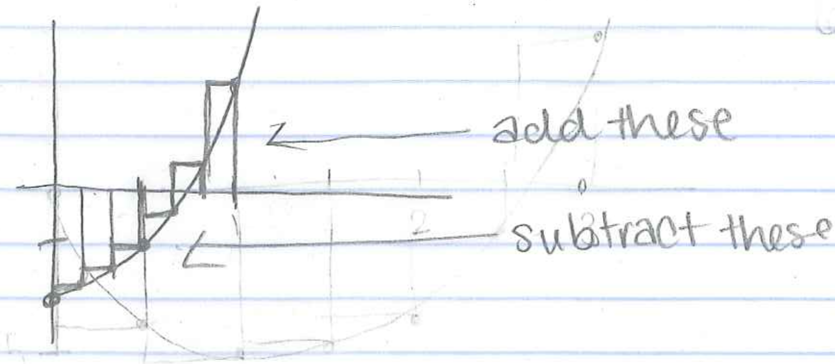
$$\int_0^4 (x-1)\,dx$$

* area under the x axis will be subtracted from area above the x-axis *

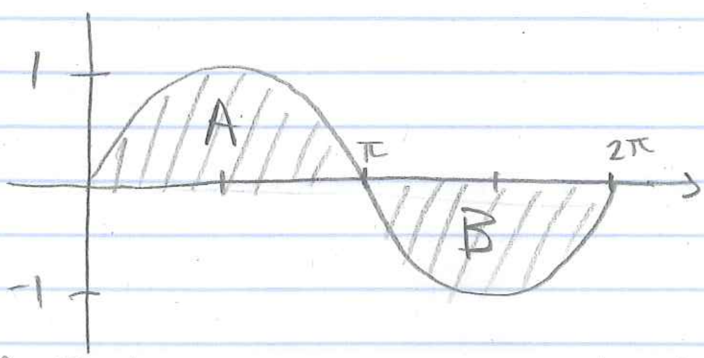
$$\frac{1}{2}(3 \cdot 3) - \frac{1}{2}(1 \cdot 1) = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$$

This is already encoded in the integral, you only have to account for it when taking the integral using the basic geometry method.

$f(x) = x^2 - 2$ on $[0,2]$ w/ 6 rect.



(4) $\int_0^{2\pi} \sin x \, dx$



A, B have same area, but

$$\int_0^{2\pi} \sin x \, dx = \text{area } A - \text{area } B$$
$$= 0$$