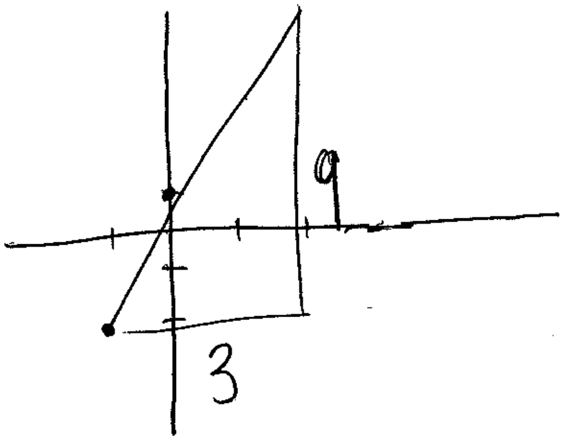
Arc Length Formula: for $f(x)$ between $x=a$, $x=b$

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Example: length of $y=3x+1$, between $x=-1$, $x=2$




$$\sqrt{3^2+9^2} = \sqrt{90} = 3\sqrt{10}$$

Arc length formula:

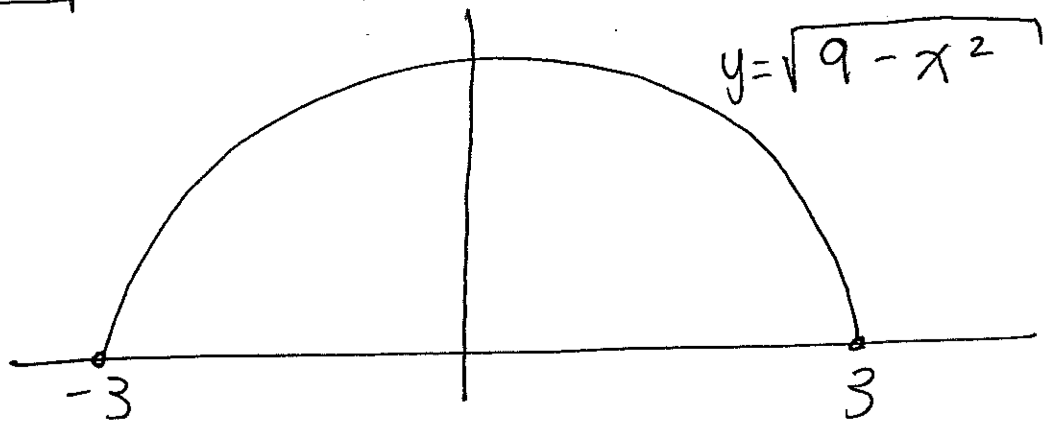
$$\int_{-1}^{2}\sqrt{1+\left(\frac{d}{dx}(3x+1)\right)^2}\,dx = \int_{-1}^{2}\sqrt{1+9}\,dx$$

$$= \int_{-1}^{2}\sqrt{10}\,dx = 2\sqrt{10}+1\sqrt{10}$$

$$= \boxed{3\sqrt{10}}$$

equal. ✓

Example: arc length of a semicircle with radius 3.




$$\text{Arc Length} = \int_{-3}^{3} \sqrt{1+\left(\frac{d}{dx}\left(\sqrt{9-x^2}\right)\right)^2}\, dx$$

$$= \int_{-3}^{3} \sqrt{1+\left(\frac{-2x}{2\sqrt{9-x^2}}\right)^2}\, dx$$

$$= \int_{-3}^{3} \sqrt{1+\frac{x^2}{9-x^2}}\, dx = \int_{-3}^{3} \sqrt{\frac{9-x^2+x^2}{9-x^2}}\, dx$$

$$= \int_{-3}^{3} \frac{3}{\sqrt{9-x^2}}\, dx$$

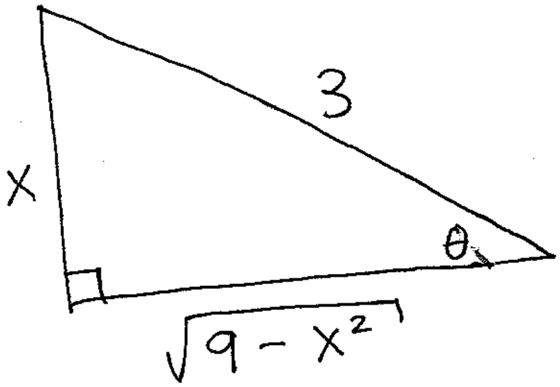
NEED TRIG SUB:

$x = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$

$$= \int_{x=-3}^{x=3} \frac{3\cdot 3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \int_{x=-3}^{x=3} \frac{9\cos\theta}{\sqrt{9\cos^2\theta}}\, d\theta$$

$$= \int_{x=-3}^{x=3} 3\cdot d\theta = 3\theta \Big|_{x=-3}^{x=3}$$




$$\frac{x}{3} = \sin\theta \Rightarrow \sin^{-1}\left(\frac{x}{3}\right) = \theta$$

$$3\theta \Big|_{x=-3}^{x=3} = 3\sin^{-1}\left(\frac{x}{3}\right)\Big|_{-3}^{3} = 3\sin^{-1}(1) - 3\sin^{-1}(-1)$$

$$= 3\left(\pi/2\right) - 3\left(-\pi/2\right)$$

$$\boxed{= 3\pi}$$

Note that this makes sense:

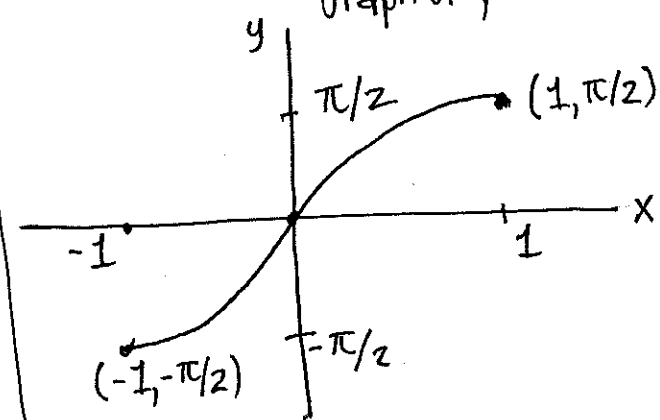
Circumference of circle $= 2\pi r$

arc length of half circle:

$$\frac{2\pi r}{2} = \underline{\pi r}$$

so when $r=3$, we get $\underline{3\pi}$

Graph of $y=\sin^{-1} x$



**Example** Find the length of the curve $y=1+3x^{3/2}$ for $0 \le x \le 1$.

$$\int_0^1 \sqrt{1+\left(\frac{d}{dx}(1+3x)^{3/2}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{3}{2}\cdot 3\sqrt{1+3x}\right)^2}\,dx$$

$$= \int_0^1 \sqrt{1+\left(\frac{9}{2}\sqrt{1+3x}\right)^2}\,dx$$

$$= \int_0^1 \sqrt{1+\frac{81}{4}+\frac{81}{4}\cdot 3x}\,dx$$

$$= \int_0^1 \sqrt{\frac{85}{4}+\frac{243}{4}\cdot x}\,dx$$

$$= \left.\frac{\frac{2}{3}\left(\frac{85}{4}+\frac{243}{4}x\right)^{3/2}}{243/4}\right|_0^1$$

$$\approx 7.074$$

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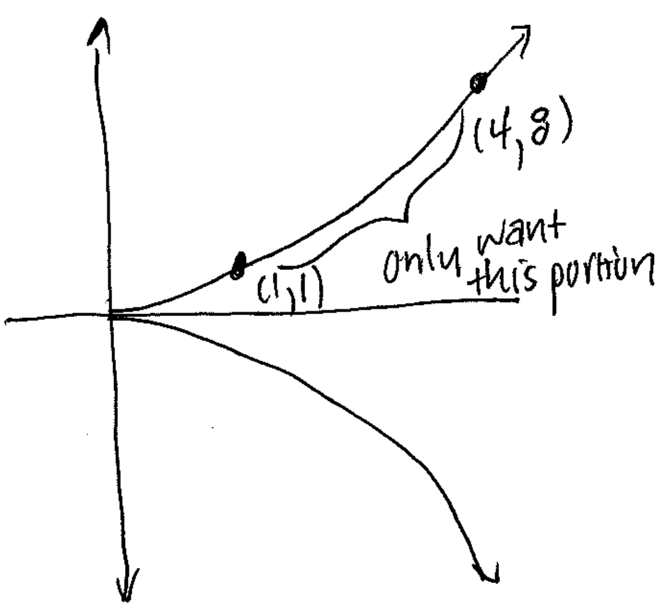
**Example** Find the length of the arc of the semicubical parabola $y^2=x^3$ between the points $(1,1)$ and $(4,8)$




$$y^2=x^3 \Rightarrow y=x^{3/2}$$

$$\text{Arc Length} = \int_1^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx$$

$$= \left.\left(\frac{2}{3}\right)\left(\frac{4}{9}\right)\left(1+\frac{9}{4}x\right)^{3/2}\right|_1^4$$

$$\approx 7.634$$