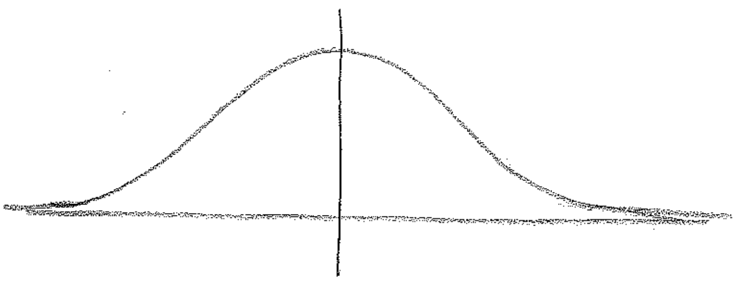
Feb. 25, 2013

Improper Integrals

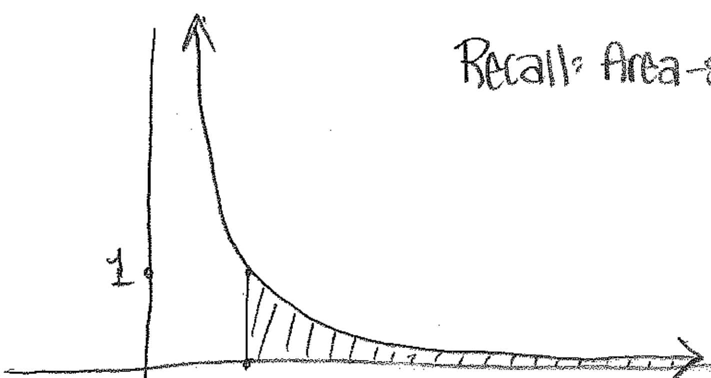
It sometimes becomes necessary to find the integral over an unbounded interval

Probability: Bell Curve



So, how could we take an integral $\int_{-\infty}^{\infty} f(x)\,dx$?

Example: Consider infinite region bounded by $y = 1/x^2$, x-axis to the right of $x=1$.



Recall: Area-so-far Function. As t gets bigger, we are accounting for more area.

$$\int_1^t \frac{1}{x^2}\,dx$$

To get the area of this infinite region, figure out what the area-so-far function equals as $t \to \infty$.

First note: $\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t = -\frac{1}{t} + 1$

Now $\lim_{t \to \infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t \to \infty} \left(-\frac{1}{t} + 1\right) = \boxed{1}$

Definition of Type 1 Improper Integrals

(a) $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ (provided this limit exists)

(b) $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$ (provided this limit exists)

(c) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$ (a is any real number)

The above integrals are called <u>convergent</u> if the corresponding limit exists and is finite and <u>divergent</u> if the limit does not exist (or is infinite).

<u>Example:</u> $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ is <u>convergent</u>
(since we showed it equals 1)

<u>Example:</u> Is $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ convergent or divergent?

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

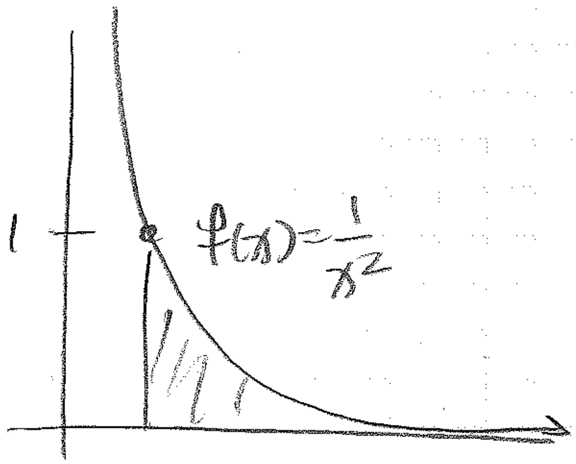
$$= \lim_{t\to\infty} \left(\ln x \Big|_1^t \right)$$

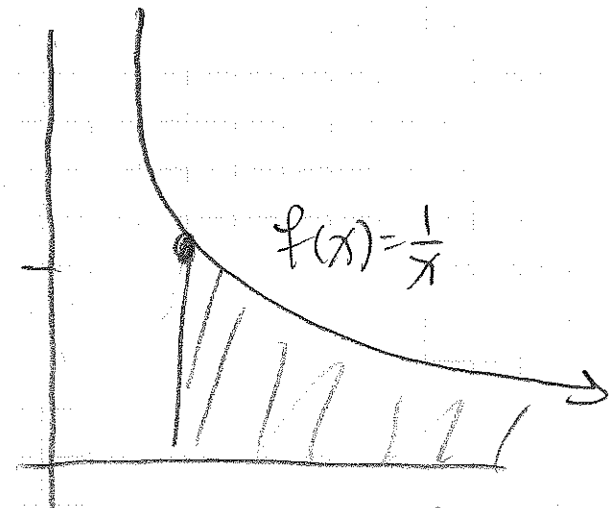
$$= \lim_{t\to\infty} (\ln t) = \text{POOF!}\ (\infty)$$

So $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ is <u>divergent</u>

Nifty Neat-O Fact:

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$ and divergent if $p<1$.





Look similar, but $1/x^2$ gets smaller faster.

Ex) $\int_{-\infty}^0 x e^x\,dx = \lim_{t\to-\infty} \int_t^0 x e^x\,dx$

Int by parts
$u=x \quad dv=e^x\,dx$
$du=dx \quad v=e^x$

$= \lim_{t\to-\infty} \left(xe^x \Big|_t^0 - \int_t^0 e^x\,dx \right)$

$= \lim_{t\to-\infty} (xe^x - e^x)\Big|_t^0$

$= \lim_{t\to-\infty} (0\cdot e^0 - e^0 - te^t + e^t) = \lim_{t\to-\infty} (-1 - te^t + \underbrace{e^t}_{\text{goes to zero}})$

What is $\lim_{t\to-\infty} te^t$? Need L'hospital's Rule.

L'hospital: If $\lim_{x\to a} f(x)/g(x)$ is of the form $0/0$ or ∞/∞ then $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$

$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} \underset{\boxed{L}}{=} \lim_{t\to-\infty} \frac{1}{-e^{-t}} = 0$

So $\int_{-\infty}^0 xe^x\,dx = -1 - 0 + 0 = \boxed{-1}$

$\underline{\text{Ex}}$ $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_0^{\infty} \frac{1}{1+x^2}\,dx + \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1}x \Big|_0^t = \lim_{t\to\infty} \tan^{-1}t - \underbrace{\tan^{-1}0}_{0} = \pi/2$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \tan^{-1}x \Big|_t^0 = \lim_{t\to-\infty} -\tan^{-1}t = \pi/2$$

So $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$

$\underline{\text{Ex}}$ (Practice) $\displaystyle\int_0^{\infty} \frac{1}{\sqrt[4]{1+x}}\,dx$

$$= \lim_{t\to\infty} \int_0^t \frac{1}{(1+x)^{1/4}}\,dx = \lim_{t\to\infty} \frac{4}{3}(1+x)^{3/4}\Big|_0^t$$

$$= \lim_{t\to\infty} \frac{4}{3}(1+t)^{3/4} - \frac{4}{3} = \text{POOF } (\infty)$$

Diverges

$\underline{\text{Ex}}$ (Practice) $\displaystyle\int_0^{\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty} \int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx$ $\quad u = x^3+1$, $du = 3x^2dx$

$$= \lim_{t\to\infty} \int_0^t \frac{1}{3}\frac{1}{\sqrt{u}}\,du$$

$$= \lim_{t\to\infty} \frac{2}{3}\sqrt{x^3+1}\Big|_0^t$$

$$= \lim_{t\to\infty} \frac{2}{3}\sqrt{t^3+1} - \frac{2}{3} \to \infty$$

Diverges.

Ex1 $\displaystyle\int_1^\infty \frac{1}{(2x+1)^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(2x+1)^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}\cdot\frac{1}{(2x+1)}\Big|_1^t$

$$= \lim_{t\to\infty} -\frac{1}{2(2t+1)} + \frac{1}{6}$$

$$= \boxed{\frac{1}{6}}$$