Feb. 20, 2013

$\int e^{-x}\cos 2x$

Announcements:
- X-hour Review tomorrow
- Midterm tomorrow (7-9, Wilder 111)
- WeBWork: 7.4 Due Monday
- HW6 Due Monday

* Finish Trig Sub Example 1

Partial Fraction Decomposition

- What is $\int \frac{2}{x-1} - \frac{1}{x+2}\,dx$?

$$= 2\ln(x-1) - \ln(x+2) + C$$

- What is $\int \frac{x+5}{x^2+x-2}\,dx$?

Hint: Combine $\frac{2}{x-1} - \frac{1}{x+2}$

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{2x+4-x+1}{x^2+x-2} = \frac{x+5}{x^2+x-2}$$

So $\int \frac{x+5}{x^2+x-2}\,dx = 2\ln(x-1) - \ln(x+2) + C$

So, we need a method to "break apart" fractions like

$$\frac{x+5}{x^2+x-2}$$

Partial Fraction Decomposition $\frac{P(x)}{Q(x)}$

- First factor the denominator.

Case 1: Denominator factors into distinct linear factors:

$$Q(x) = (a_1x - b_1)(a_2x - b_2)\cdots(a_kx - b_k)$$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x - b_1} + \frac{A_2}{a_2x - b_2} + \cdots + \frac{A_k}{a_kx - b_k}$$

$A_1, A_2, \ldots, A_k$ are just constants. (NO x-term)

Example:

- Setup:

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{\underbrace{(x+2)(x-1)}_{\text{distinct linear factors}}} = \frac{A}{x+2} + \frac{B}{x-1}$$

- solve for unknowns (A, B)

$$(x+2)(x+1)\left(\frac{x+5}{x^2+x-2}\right) = \left(\frac{A}{x+2} + \frac{B}{x-1}\right)(x+2)(x-1)$$

$$x+5 = A(x-1) + B(x+2)$$

$$x+5 = \underbrace{Ax + Bx}_{=x} \underbrace{- A + 2B}_{=5}$$

$$\begin{cases} Ax + Bx = x \\ -A + 2B = 5 \end{cases} \Rightarrow \begin{cases} A + B = 1 \\ -A + 2B = 5 \end{cases}$$

- solve the system of equations

$$3B = 6 \Rightarrow B = 2$$

$$A + B = A + 2 = 1 \Rightarrow A = -1$$

$$\frac{x+5}{x^2+x-2} = \frac{-1}{x+2} + \frac{2}{x-1}$$

$$\boxed{\begin{array}{l}\underline{\text{Case 2}}\text{: Denominator factors into linear factors, some of which are repeated} \\ \underline{\text{Fact:}} \\ \dfrac{P(x)}{(ax-b)^r} = \dfrac{A_1}{ax-b} + \dfrac{A_2}{(ax-b)^2} + \dfrac{A_3}{(ax-b)^3} + \dots + \dfrac{A_r}{(ax-b)^r} \\ \\ \text{* use this fact w/ the strategy in case 1 *}\end{array}}$$

<u>Example</u>: $$\frac{x^3-x+1}{x^2(x-1)^3(x+2)} = \underbrace{\frac{A}{x} + \frac{B}{x^2}}_{x^2} + \underbrace{\frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}}_{(x-1)^3} + \underbrace{\frac{F}{x+2}}_{(x+2)}$$

A, B, C, D, E, F are constants

<u>Example</u>: $$\frac{x+1}{x^2+6x+9} = \frac{x}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

Solve for A, B: $$\frac{x}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2}$$

$$x = A(x+3) + B$$

$Ax = x \Rightarrow A = 1$

$3A + B = 1$

$3 + B = 1 \Rightarrow B = -2$

$$\frac{x}{x^2+6x+9} = \frac{1}{x+3} - \frac{2}{(x+3)^2}$$

Back to integrals:

$$\int \frac{x+1}{x^2+6x+9}\,dx = \int \frac{1}{x+3} - \frac{2}{(x+3)^2}\,dx = \ln(x+3) - 2\int (x+3)^{-2}\,dx$$

$$= \boxed{\ln(x+3) + \frac{2}{(x+3)} + C}$$

The Process:

(1) Factor denominator
(2) Write as sum of fractions
(3) Solve for unknowns

Limitations

- degree of denominator must be greater than the degree of the numerator
- denominator factors into linear factors

Examples:

(1) $\displaystyle\int \frac{10x^2+2}{4x^3-4x^2+x}\,dx$

Factor denominator:

$$4x^3-4x^2+x = x(4x^2-4x+1)$$
$$= x(2x-1)^2$$

Write as sum of fractions:

$$\frac{10x^2+2}{x(2x-1)^2} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$

$$10x^2+2 = A(2x-1)^2 + Bx(2x-1) + Cx$$
$$10x^2+2 = A\cdot 4x^2 - A\cdot 4x + A + B\cdot 2x^2 - B\cdot x + Cx$$

$$\begin{aligned} 10 &= 4A + 2B \\ 0 &= -4A - B + C \\ 2 &= A \end{aligned} \Rightarrow \begin{aligned} &10 = 4(2) + 2B \Rightarrow B = 1 \\ &0 = -4(2) - (1) + C \Rightarrow C = 9 \end{aligned}$$

$$\frac{10x^2+2}{x(2x-1)^2} = \frac{2}{x} + \frac{1}{2x-1} + \frac{9}{(2x-1)^2}$$

$$\int\left(\frac{2}{x} + \frac{1}{2x-1} + \frac{9}{(2x-1)^2}\right)dx = 2\ln x + \frac{\ln|2x-1|}{2} - \frac{9}{2(2x-1)} + C$$