Feb. 13, 2013

Announcements: HW5 Due Monday
Midterm next week
WW: 7.2 Due Fri
7.3 Due Wed (2/20)

Evaluating $\int \sin^m x \cos^n x \, dx$

(a) Power of Cosine is odd
- Save one cosine factor
- use $1-\sin^2 x = \cos^2 x$ to change everything else into sine.
- Substitute $u = \sin x$.

(b) Power of sine is odd
- save one sine factor
- use $1-\cos^2 x = \sin^2 x$ to change everything else into cosine.
- Substitute $u = \cos x$

(c) Both sine & cosine have even powers
- use the identities (half angle formulas)

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\sin 2x)$$

(less often) $\sin x \cos x = \frac{1}{2}\sin 2x$

Examples of property (c):

(1) $\int_0^\pi \sin^2 x \, dx = \int_0^\pi \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^\pi$

use half-angle formulas

$= \frac{1}{2}\left(\pi - \frac{0}{2} - 0 + \frac{0}{2}\right)$

$= \boxed{\pi/2}$

(2) $\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$

$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$= \frac{1}{4}\int \Big(1 - 2\cos 2x + \underbrace{\cos^2 2x}_{\text{even power. Use half-angle again.}}\Big)\,dx$

$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\sin 2(2x))\right)dx$

$= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\sin 4x \, dx$

$$= \frac{1}{4}\left(\frac{3}{2}x - \frac{2\sin 2x}{2} + \frac{(-\cos 4x)}{8}\right) + C$$

$$= \frac{3}{8}x - \frac{\sin 2x}{4} - \frac{\cos 4x}{32} + C$$

More Examples: (Harder)

- $\int \frac{\sin^3\sqrt{x}}{\sqrt{x}}\,dx$ • use substitution first then use trig techniques

$u=\sqrt{x}$ $du = \frac{1}{2\sqrt{x}}dx$

$$\int \frac{\sin^3\sqrt{x}}{\sqrt{x}}\,dx = \int 2\cdot \sin^3 u\,du = \int 2\sin u\cdot \sin^2 u\,du$$

$$= \int 2\sin u\cdot(1-\cos^2 u)\,du$$

sub again: $w = \cos u$
$dw = -\sin u\,du$

$$= \int -2(1-w^2)\,dw$$

$$= -2(w - w^3/3) + C$$

$$= -2\left(\cos u - \frac{\cos^3 u}{3}\right) + C$$

$$= -2\left(\cos\sqrt{x} - \frac{\cos^3\sqrt{x}}{3}\right) + C$$

- $\int \sin^2(3x)\cos^5(3x)\,dx = \int \sin^2(3x)(1-\sin^2(3x))^2\cdot\cos(3x)\,dx$

$$= \int \sin^2(3x)(1-2\sin^2(3x)+\sin^4(3x))\cdot\cos(3x)\,dx$$

$u=\sin(3x)$ $du = 3\cos(3x)\,dx$

$$= \int \frac{1}{3}u^2(1-2u^2+u^4)\,du = \frac{1}{3}\int u^2 - 2u^4 + u^6\,du$$

$$= \frac{1}{3}(u^3/3 - 2u^5/5 + u^7/7) + C = \frac{1}{3}\left(\frac{\sin^3(3x)}{3} - \frac{2\cdot\sin^5(3x)}{5} + \frac{\sin^7(3x)}{7}\right) + C$$

There is a similar process for integrating

$$\int \tan^m x \sec^n x \, dx$$

Recall the identities:

$1+\tan^2 x = \sec^2 x \qquad \tan^2 x = \sec^2 x - 1$

$(1+\cot^2 x = \csc^2 x \qquad \cot^2 x = \csc^2 x - 1)$

Also:

$\frac{d}{dx} \tan x = \sec^2 x \qquad \left(\frac{d}{dx} \cot x = -\csc^2 x \right.$

$\frac{d}{dx} \sec x = \sec x \tan x \qquad \left. \frac{d}{dx} \csc x = \right.$

we will either save $\sec^2 x$ OR $\sec x \tan x$

Examples:

(1) $\int \tan^6 x \sec^4 x \, dx = \int \sec^2 x (\tan^6 x)(\sec^2 x) \, dx$

$= \int \sec^2 x (\tan^6 x)(1+\tan^2 x) \, dx$

$u = \tan x \quad du = \sec^2 x \, dx$

$= \int u^6 (1+u^2) \, du = \int u^6 + u^8 \, du$

$= \frac{u^7}{7} + \frac{u^9}{9} + C$

$= \boxed{\frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C}$

(2) $\int \tan^5 \theta \sec^7 \theta \, d\theta$ *save $\sec\theta \tan\theta$ term*

$= \int \sec\theta \tan\theta (\tan^4 \theta \sec^6 \theta) \, d\theta$

$= \int \sec\theta \tan\theta ((\tan^2 \theta)^2 \sec^6 \theta) \, d\theta =$

$$\int \sec\theta\tan\theta\left((\sec^2\theta-1)^2\cdot\sec^6\theta\right)d\theta$$

$u=\sec\theta \quad du=\sec\theta\tan\theta\ d$

$$=\int (u^2-1)^2u^6\,du=\int (u^4-2u^2+1)u^6\,du=\int u^{10}-2u^8+u^6\,du$$

$$=\frac{u^{11}}{11}-2\frac{u^9}{9}+\frac{u^7}{7}+C$$

$$=\frac{\sec^{11}\theta}{11}-2\frac{\sec^9\theta}{9}+\frac{\sec^7\theta}{7}+C$$

Evaluating $\int \tan^m x\sec^n x\,dx$

(a) Power of secant is even
- save $\sec^2 x$
- use $\sec^2 x = 1+\tan^2 x$ to change everything else into $\tan x$
- substitute $u=\tan x$

(b) Power of tangent is odd
- save $\sec x\tan x$
- use $\tan^2 x=\sec^2 x-1$ to change everything else into $\sec x$
- substitute $u=\sec x$

Practice:

(1) $\int \sec^6(\pi x)\tan^4(\pi x)\,dx=\int \sec^2(\pi x)(\sec^2(\pi x))^2\tan^4(\pi x)\,dx$

$$=\int \sec^2(\pi x)(1+\tan^2(\pi x))^2\tan^4(\pi x)\,dx$$

$u=\tan(\pi x) \quad du=\pi\sec^2(\pi x)\,dx$

$$=\frac{1}{\pi}\int (1+u^2)^2u^4\,du=\frac{1}{\pi}\int (1+2u^2+u^4)u^4\,du$$

$$=\frac{1}{\pi}\int u^4+2u^6+u^8\,du=\frac{1}{\pi}\left(\frac{u^5}{5}+\frac{2u^7}{7}+\frac{u^9}{9}\right)+C$$