Integration by Parts

Use integration by parts (in conjuction with other integration techniques you already know) to solve the following integrals:

1. $\int_0^{\pi/6} x \cdot \cos(3x)\, dx$ $= x\frac{\sin 3x}{3}\Big|_0^{\pi/6} - \int_0^{\pi/6} \frac{\sin 3x}{3} dx = \frac{x \sin 3x}{3} + \frac{\cos 3x}{9}\Big|_0^{\pi/6}$

$u = x \quad dv = \cos 3x\, dx$

$du = dx \quad v = \frac{\sin 3x}{3}$

$= \frac{\frac{\pi}{6}\cdot \sin \pi/2}{3} + \frac{\cos \pi/2}{9} - \frac{\cos 0}{9} = \boxed{\frac{\pi}{18} - \frac{1}{9}}$

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2. $\int_1^2 x^2 \cdot \ln x\, dx$ $= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{3x} dx = \frac{x^3}{3}\ln x - \frac{x^3}{9}\Big|_1^2$

$u = \ln x \quad dv = x^2 dx$

$du = \frac{dx}{x} \quad v = \frac{x^3}{3}$

$= \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9} = \boxed{\frac{8}{3}\ln 2 - \frac{7}{9}}$

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3. $\int_0^{\pi/2} (x^2 + 2x) \cdot \cos x\, dx$ $= (x^2+2x)\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} (2x+2)\sin x\, dx$ $= (x^2+2x)\sin x + (2x+2)\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} 2\cos x\, dx$

$u = x^2+2x \quad dv = \cos x\, dx$

$du = (2x+2)dx \quad v = \sin x$

$u = 2x+2 \quad dv = \sin x\, dx$

$du = 2dx \quad v = -\cos x$

$= (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x\Big|_0^{\pi/2}$

$= \left(\frac{\pi}{2}\right)^2 + \pi - 2 - 2$

$= \boxed{\frac{\pi^2}{4} - 4}$

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4. $\int_0^1 \tan^{-1} x\, dx$ $= x\tan^{-1}x\Big|_0^1 - \underbrace{\int_0^1 \frac{x}{1+x^2} dx}_{\text{u-substitution}} = x\tan^{-1}x\Big|_0^1 - \int_{x=0}^{x=1} \frac{1}{2u} du = x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\ln(1+x^2)\Big|_0^1$

$u = \tan^{-1} x \quad dv = dx$

$du = \frac{1}{1+x^2} dx \quad v = x$

u-substitution

$u = 1+x^2 \quad du = 2x\, dx$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}$

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5. $\int \cos x \cdot \ln(\sin x)\, dx$ $= \int \ln(u)\, du$ $= u\ln u - \int du = u \ln u - u + C = \boxed{\sin x \ln(\sin x) - \sin x + C}$

Use substitution first:

$u = \sin x \quad du = \cos x\, dx$

Integration by parts:

$w = \ln u \quad dv = 1\, du$

$dw = \frac{1}{u} du \quad v = u$

Sub back in

Feb. 11, 2013

Announcements:

Integration by Parts: Worksheet

Note: $\sin^3 x = \sin x \cdot \sin x \cdot \sin x$

etc. for other powers + trig functions

Trigonometric Integrals

We can do: $\int \cos x \sin^5 x \, dx \quad u = \sin x \quad du = \cos x \, dx$

$$= \int u^5 \, du = \frac{u^6}{6} + C = \frac{\sin^6 x}{6} + C$$

What about $\int \cos^3 x \sin^5 x \, dx$

substitution right away won't cut it.

For sub. it works best for there to be only one term of $\cos x$.

Let's do it:

Recall: $\cos^2 x = 1 - \sin^2 x$

So $\int \cos^3 x \sin^5 x \, dx = \int \cos x (1 - \sin^2 x) \sin^5 x \, dx$

$$= \int \cos x (\sin^5 x - \sin^7 x) dx$$

Now, we can sub: $u = \sin x \quad du = \cos x \, dx$

$$= \int u^5 - u^7 \, du = \frac{u^6}{6} - \frac{u^8}{8} + C$$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

Ex: $\int \sin^3 x \, dx = \int \sin x (\sin^2 x) dx = \int \sin x (1 - \cos^2 x) \, dx$

$u = \cos x \quad du = -\sin x \, dx$

$$= \int -(1 - u^2) du = -u + u^3/3 + C$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

The identities to remember: $1 - \sin^2 x = \cos^2 x$

$1 - \cos^2 x = \sin^2 x$

Evaluating $\int \sin^m x \cos^n x \, dx$

(a) Power of Cosine is odd
- Save one cosine factor
- use $1-\sin^2 x = \cos^2 x$ to change everything else into sine.
- Substitute $u = \sin x$.

(b) Power of sine is odd
- save one sine factor
- use $1-\cos^2 x = \sin^2 x$ to change everything else into cosine.
- Substitute $u = \cos x$

(c) Both sine & cosine have even powers
- use the identities (half angle formulas)

$\sin^2 x = \frac{1}{2}(1-\cos 2x) \quad \cos^2 x = \frac{1}{2}(1+\sin 2x)$

(less often) $\sin x \cos x = \frac{1}{2}\sin 2x$

Examples of property (c):

(1) $\int_0^\pi \sin^2 x \, dx = \int_0^\pi \frac{1}{2}(1-\cos 2x)\, dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^\pi$

use half-angle formulas

$= \frac{1}{2}\left(\pi - \frac{0}{2} - 0 + \frac{0}{2}\right)$

$= \boxed{\pi/2}$

(2) $\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$

$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

even power. Use half-angle again.

$= \frac{1}{4}\int (1 - 2\cos 2x + \underbrace{\cos^2 2x}) \, dx$

$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\sin 2(2x))\right) dx$

$= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\sin 4x \, dx$