Arclength: continued from Wed 2/27.

Now we want to write the $\lim_{n\to\infty} \sum_{i=1}^{n}$ as an integral:

$$P_i = (x_i, f(x_i)) \qquad P_{i-1} = (x_{i-1}, f(x_{i-1}))$$

$$d(P_i, P_{i-1}) = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

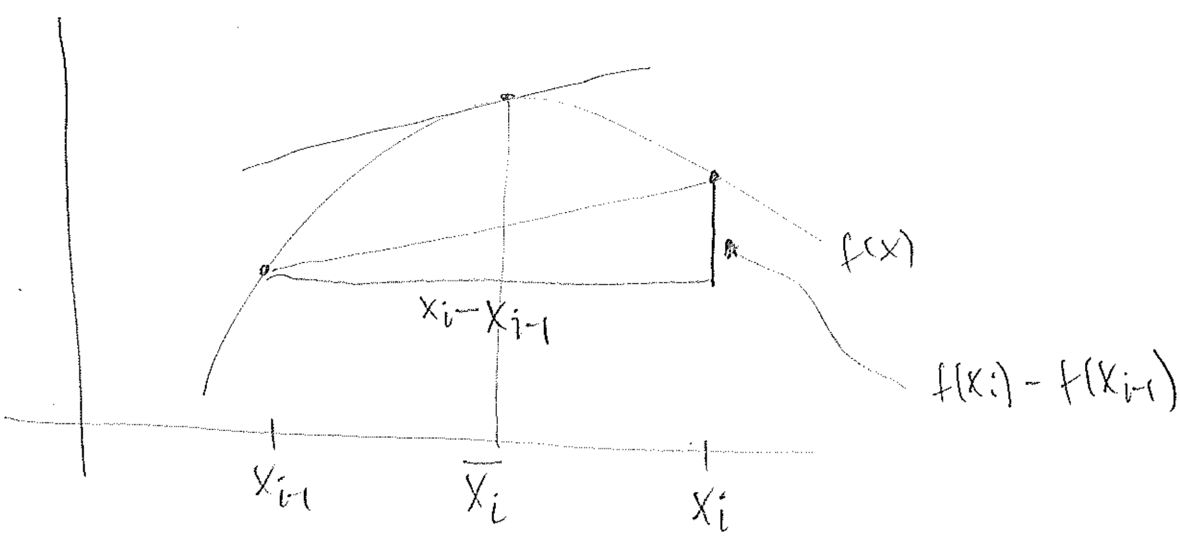
we can rewrite $x_i - x_{i-1} = \Delta x$

to rewrite $f(x_i) - f(x_{i-1})$ we use Mean Value Thm applied to the function $f(x)$ on the interval $[x_{i-1}, x_i]$

MVThm: there exists $\bar{x}_i \in (x_{i-1}, x_i)$ i.e. $x_{i-1} < \bar{x}_i < x_i$

such that $f'(\bar{x}_i) = \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$ equivalently

$$f'(\bar{x}_i) \cdot (x_i - x_{i-1}) = f(x_i) - f(x_{i-1})$$



Now we can rewrite $f(x_i) - f(x_{i-1}) = f'(\bar{x}_i) \cdot \Delta x$

and $d(P_i, P_{i-1}) = \sqrt{(\Delta x)^2 + [f'(\bar{x}_i) \cdot \Delta x]^2}$

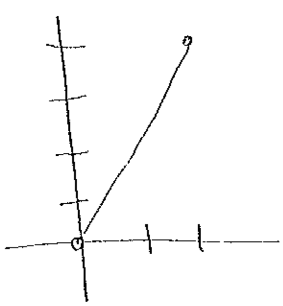
$= \sqrt{(\Delta x)^2 + [f'(\bar{x}_i)]^2 (\Delta x)^2} = \sqrt{(\Delta x)^2 \left[1 + [f'(\bar{x}_i)]^2\right]}$

$= \Delta x \sqrt{1 + [f'(\bar{x}_i)]^2}$

Thus Arc length $= \lim\limits_{n \to \infty} \sum_{i=1}^{n} d(P_i, P_{i-1}) = \lim\limits_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(\bar{x}_i)]^2} \cdot \Delta x$

$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Ex] Find arc length of the curve $f(x) = 2x$ between $(0,0)$ and $(2,4)$

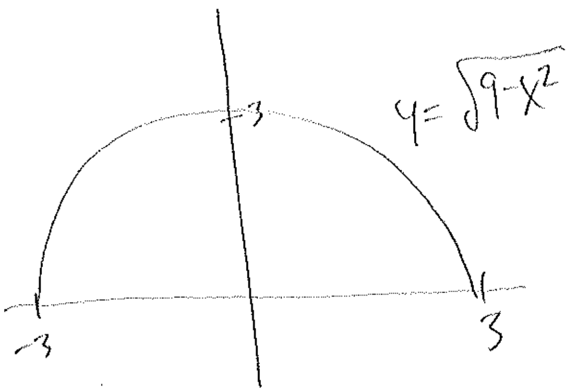


$f'(x) = 2$

arc length $= \int_0^2 \sqrt{1 + (2)^2}\, dx = \int_0^2 \sqrt{5}\, dx$

$= x \cdot \sqrt{5} \Big|_0^2 = 2\sqrt{5} - 0 = \sqrt{20}$

Motivating Example : Find length of the curve $y = \sqrt{9-x^2}$ between $(-3,0)$ and $(3,0)$



$$\text{length} = \frac{2\pi r}{2} = \pi r$$

$$y = (9-x^2)^{1/2} \qquad y' = \tfrac{1}{2}(9-x^2)^{-1/2}(2x) = \frac{x}{\sqrt{9-x^2}}$$

$$[y']^2 = \frac{x^2}{9-x^2}$$

$$\text{Arc Length} = \int_{-3}^{3} \sqrt{1 + \frac{x^2}{9-x^2}}\, dx = \int_{-3}^{3} \sqrt{\frac{9-x^2+x^2}{9-x^2}}\, dx$$

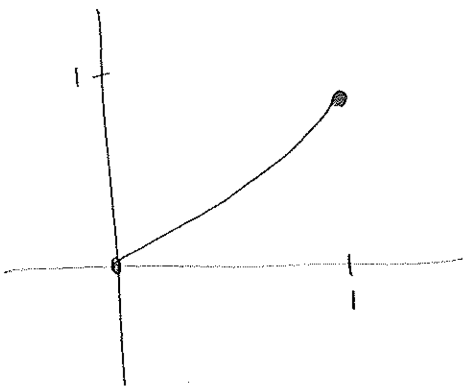
$$= \int_{-3}^{3} \sqrt{\frac{9}{9-x^2}}\, dx = \int_{-3}^{3} \frac{3}{\sqrt{9-x^2}}\, dx \qquad \begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\, d\theta \end{array}$$

$$= \int_{-3}^{3} \frac{3}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\, d\theta = \int_{-3}^{3} \frac{3}{3\cos\theta} \cdot 3\cos\theta\, d\theta = 3\theta \Big|_{x=-3}^{x=3}$$

$$= 3\arcsin\left(\frac{x}{3}\right)\Big|_{-3}^{3} = 3\cdot\arcsin(1) - 3\cdot\arcsin(-1)$$

$$= 3\cdot\frac{\pi}{2} - 3\left(-\frac{\pi}{2}\right) = 3\pi$$

ex1 find length of $y = \frac{2}{3}x^{3/2}$, $0 \le x \le 1$



$y' = x^{1/2}$ $\qquad (y')^2 = x$

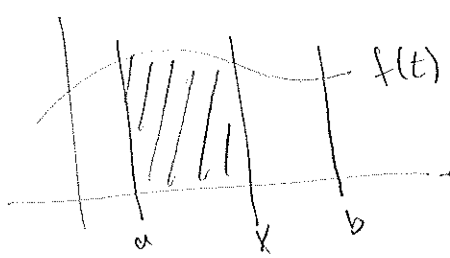
length $= \int_0^1 \sqrt{1+x}\, dx$

$= (1+x)^{3/2} \cdot \frac{2}{3} \Big|_0^1 = 2^{3/2}\left(\frac{2}{3}\right) - 1 \cdot \frac{2}{3} \approx 1.2$

$\boxed{\text{Mon } 3/4}$

Arc Length Function : "arc length so far"

Recall: Area so far : $g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$



$$S(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt \qquad a \le x \le b$$

gives the arc length of the curve $f(t)$ from initial point $(a, f(a))$ to another point $(x, f(x))$

Arc Length Contest

An interesting property of arc length is that it is **not** dependent on the area under a curve.
Your task is to find a curve that fits the following requirements:

1. $f(0) = 0$ and $f(2) = 0$
2. $f(x) \geq 0$ for $0 \leq x \leq 2$
3. The are under the graph of f from 0 to 2 is equal to 2.

Your challenge is to find a curve with the **shortest** possible arc length you can manage. The team or individual that comes up with the curve that has the shortest arc length that meets all the requirements above will be the winner!

A similar challenge can be found on page 545 of Stewart. Use this if you would like to see some pictures. Note that their requirements are different than the requirements above. You must follow the requirements on this sheet or you will be disqualified.

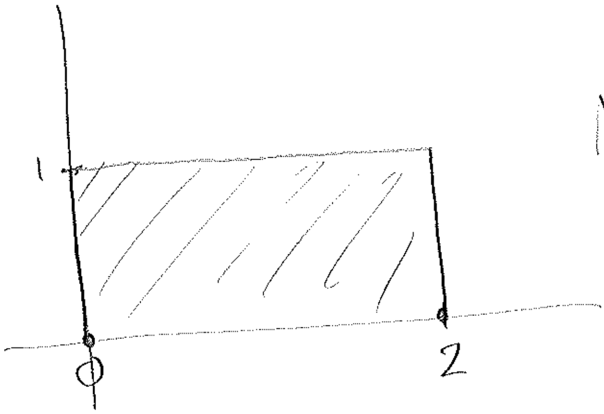
The Rules

- You may work as an individual or in a team of 2. Larger teams will not be accepted
- You may use **NO** outside resources to complete this contest. The only resources you may draw upon is your textbook, your notes, and the Wolfram Alpha calculator (or your own calculator) to find the arc length of your possibilities
- You may not consult the tutors or your professor (except for clarification). You may not consult other teams (nor should you want to - see prize below)
- Submit your entry in writing to your professor by the last day of class (March 8). Entries must be **legible** and coherent. You should include the function and computations that show the function meets all three criterion. Make sure both team names are on your entry (if applicable).

The Prize

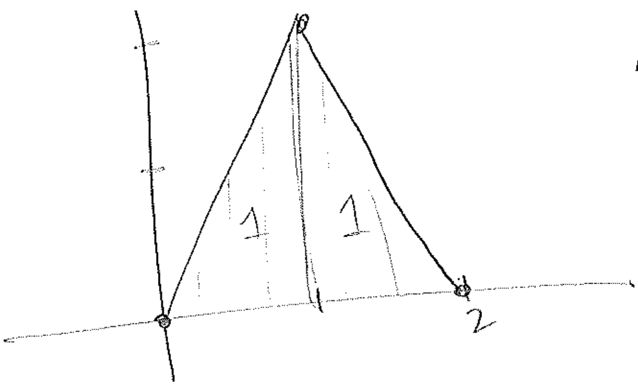
Each member of the winning team will have 5 pts added to their final exam score. If there is a tie, the points will be split between teams (i.e. for a 2-way tie concerned parties will get 2.5 points added to their final exam).

Arc length Contest



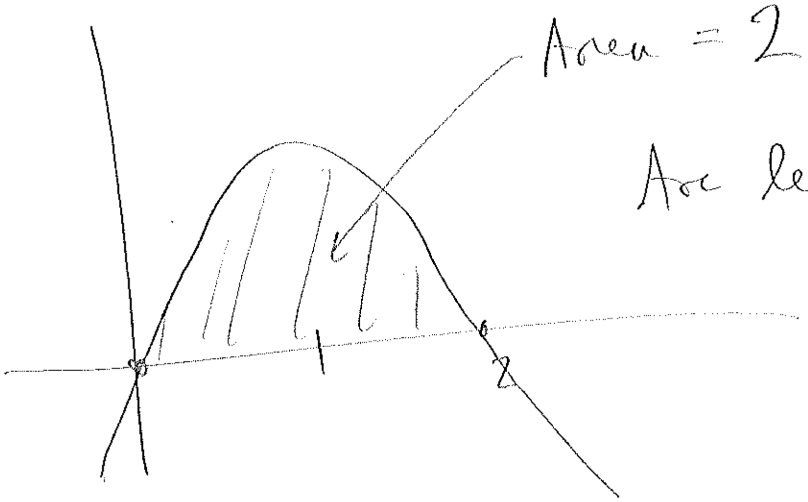
Area = 2

Arc length = 4



Area = 2

Arc length = $2\sqrt{5} \approx 4.47$



Area = 2

Arc length = ?